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Analysis of Gutenberg-Richter *b*-value and m_{max} Part II: Estimators for *b*-value

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Abstract

This report is the second of a series of three which have the main goal to achieve a method to estimate the parameters b and $m_{\max}$ that are essential when Gutenberg-Richter law is used for seismic hazard assessment. This paper is devoted to analyze the estimators of b-value.
We give an estimator for the expected value of $M_{(N)}$ and an exact and numerically stable variance for it; $M_{(N)}$ represents the ordered values (by size) of magnitude catalogue.
We go on the development of the theory of Kijko-Sellevoll functions, applying them to calculate generalized b estimators of the seismic catalogue.

Keywords: $m_{\max}$, b-value, Gutenberg-Ritcher distribution function - Kijko Sellevoll estimator

Resumen

Este artículo es el segundo de una serie de tres, que tienen como objetivo principal obtener un método para calcular los parámetros b y $m_{\max}$ que son fundamentales cuando se utiliza la ley de Gutenberg – Richter para la estimación de la peligrosidad sísmica. En este trabajo en particular, nos enfocamos al análisis de estimadores del valor b.
Proponemos un estimador para el valor esperado de $M_{(N)}$ y una expresión exacta y numéricamente estable para su varianza; $M_{(N)}$ representa los valores ordenados de las magnitudes del catálogo sísmico.
Continuamos con el desarrollo de la teoría de las funciones de Kijko-Sellevoll, aplicándolas al cálculo de estimadores de b generales.

Palabras clave: $m_{\max}$ - b - función de distribución de Gutenberg-Ritcher - estimador de Kijko Sellevoll

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Introduction

We are analyzing the double truncated Gutenberg-Richter distribution function

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$$f(m) = \frac{\beta \exp\left[-\beta\left(m - m_{\min}\right)\right]}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]},$$

which has cumulative distribution function (CDF)

$$F_M\left(m \mid m_{\max}\right) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \dfrac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]} & \text{for } m_{\min} \le m < m_{\max}, \\ 1, & \text{for } m_{\max} \le m. \end{cases} \tag{1}$$

The limit distribution function was given as

$$F_M\left(m \mid m_{\max} = \infty\right) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ 1-\exp\left[-\beta\left(m-m_{\min}\right)\right] & \text{for } m_{\min} \le m, \end{cases}$$

and

$$F_M\left(m \mid m_{\max} = m_{\min}\right) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ 1, & \text{for } m_{\min} \le m, \end{cases}$$

where $\beta = b\log(10)$, $m_{\min}$ is a threshold magnitude and $m_{\max}$ is a maximum possible magnitude. We could assume that $m_{\max} \in \left[m_{\min}, \infty\right]$. If we let $M_1,\ M_2,\ \dots,\ M_N \in \left[m_{\min}, m_{\max}\right]$ be a set of random variables from the catalogue C_N of size N, and we let $M_{(1)} \le M_{(2)} \le \cdots \le M_{(N)}$ denote the ordered values of $M_1,\ M_2, \dots,\ M_N$, so the random variable $M_{(N)}$ is a maximum in the catalogue C_N. We assume that these random variables are independently and identically distributed (iid) with CDF of $F_M(m)$ given by (1). Let now $m_{(1)} \le m_{(2)} \le \cdots \le m_{(N)}$ to be an ordered sample of magnitudes, where $m_{(1)}$ is a minimum observed magnitude ($m_{\min} \le m_{(1)}$) and $m_{(N)}$ is a maximum observed magnitude ($m_{(N)} \le m_{\max}$), having a CDF

$$F_{M_{(n)}}\left(m \mid m_{\max}\right) = \begin{cases} 0, & \text{for } m < m_{\min}, \\ \left[F_M\left(m \mid m_{\max}\right)\right]^n & \text{for } m_{\min} \le m < m_{\max}, \\ 1, & \text{for } m_{\max} \le m. \end{cases} \tag{2}$$

We showed in a previous paper (Haarala and Orosco, 2016) that using a Kijko-Sellevoll function 1 (KS-1)

$$f_n^{KS-1}(x) = \sum_{k=1}^{\infty} \frac{(1-\exp[-x])^k}{k+n}$$

or a Kijko-Sellevoll function 2 (KS-2)

$$f_n^{KS-2}(x) = n\sum_{k=1}^{\infty} \frac{(1-\exp[-x])^k}{k(k+n)}$$

we can write the expected value of the maximum $M_{(n)}$ of the catalogue as

$$\begin{aligned} E\left(M_{(n)} \mid m_{\max}\right) &= m_{\max} - \frac{1}{\beta} f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) \\ &= m_{\min} + \frac{1}{\beta} f_n^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right). \end{aligned} \tag{3}$$

The relation between KS-1 and KS-2 functions is

$$f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) + f_n^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right) = \beta\left(m_{\max}-m_{\min}\right). \tag{4}$$

Estimator for expected value $M_{(N)}$

Suppose that we have a set of iid events $m_1,\ldots,m_N$ from the catalogue C_N. We can divide C_N into N sub-catalogues such that each sub-catalogue $C_{1:k}$ (where $k=1,\ldots,N$) has one and only one event. Each of them makes a catalogue of size $n=1$ with maximum observed magnitude $m_{(1):k}$. Because of each catalogue has only one event, $m_{(1):k} = m_k$. The mean value of maximum observed values is the unbiased estimator for the expected value $M_{(1)}$, thus

$$\hat{E}\left(M_{(1)} \mid m_{\max}\right) = \frac{m_{(1):1} + \cdots + m_{(1):N}}{N} = \overline{m}_{(1)} = \frac{m_1 + \cdots + m_N}{N} = \overline{m}. \tag{5}$$

This shows that the mean value of the maximums of the catalogues $C_{1:k}$ is an estimator for the expected value of $M_{(1)}$ ($M_{(1)}$ is a minimum but also a maximum since each catalogue $C_{1:k}$ has only one event). In the same way we could create the estimator for sub-catalogues $C_{2:k}$ (size $n=2$; each sub-catalogue has two and only two events)

$$\hat{E}\left(M_{(2)} \mid m_{\max}\right) = \frac{m_{(2):1} + \cdots + m_{(2):\lfloor N/2\rfloor}}{\lfloor N/2\rfloor} = \overline{m}_{(2)},$$

where $\lfloor\cdot\rfloor$ is a floor function (maximum integer value m such that $m \le N/2$). If the size of catalogue N is odd, then there is a value which does not belong to any sub-catalogue and we can choose it randomly.

In general

$$\hat{E}\left(M_{(n)} \mid m_{\max}\right) = \frac{m_{(n):1} + \cdots + m_{(n):\lfloor N/n \rfloor}}{\lfloor N/n \rfloor} = \bar{m}_{(n)}, \tag{6}$$

where $1 \le n \le N$ and $m_{(n):k}$ is a maximum of the sub-catalogue $C_{n:k}$ $(1 \le k \le \lfloor N/n \rfloor)$, which has n and only n events. The estimator in the case is $n = N$ is

$$\hat{E}\left(M_{(N)} \mid m_{\max}\right) = \frac{m_{(N):1}}{1} = \bar{m}_{(N)} = m_{(N)}. \tag{7}$$

The mean value from one event is the event itself. This shows also that the maximum observed value is the unbiased estimator for the expected value of maximum of the catalogue C_N. Pisarenko et al (1996) showed that (7) is the best unbiased estimator and equation (6) yields to (7) when the sub-catalogue size is the same than catalogue size itself.

In the case that $n = 1$, each sub-catalogue has only one event which is also the maximum so we will always get the same mean, independently the way we select the events from the catalogue.

When $n = N$, we have only one sub-catalogue (which has the same events than the catalogue), so we will have always the same maximum, indistinctly the order we take the events from catalogue to conform the sub-catalogue,

In the cases $1 < n < N$ the situation is different. For example, if we want to have sub-catalogue $N-1$, we choose randomly the events from the catalogue C_N. It could happen that the sub-catalogue $C_{N-1:1}$ has or has not the event $m_{(N)}$. If the event belongs to the sub-catalogue, its estimator for the maximum is $m_{(N)}$, other way it is $m_{(N-1)}$. This situation will present for all n which is not 1 or N. This is the reason for choosing the events randomly to create the sub-catalogues. Of course the simulated catalogue is random and it is not necessarily to randomize the events, but randomizing the selected events we can get different values for the estimator. Real catalogues can have some systematic changes of the β-value (see for example Cao and Gao, 2002), and to avoid the bias, what can happen by rejecting systematically the last terms of the catalogue, it is better to choose the sub-catalogues randomly.

In a previous paper (Haarala and Orosco, 2016) we used the same technique to find an estimator for the expected value. There we generated 1000 artificial catalogues of size n. From each one we took the maximum and calculated the mean of them. In that case we had unbounded number of total events. In the case of estimator (6) the total number of events is bounded (like it is in earthquake catalogue) and the number of the maximums in the mean value depends upon the number of the sub-catalogues, what is possible to get from those limited number of events.

Estimators for β -value

In the case we consider the sub-catalogues of size $n=1$ we can write the expected values (3) as

$$\overline{m}_{(1)} = \overline{m} = m_{\max} - \frac{1}{\beta} f_1^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right)$$

in case KS-1 and

$$\overline{m}_{(1)} = \overline{m} = m_{\min} + \frac{1}{\beta} f_1^{KS-2}\left(\beta\left(m_{\max} - m_{\min}\right)\right).$$

in case KS-2. The KS-1 function can write in the form of a partial sum of $n=1$ (Haarala and Orosco, 2016)

$$\begin{aligned} \overline{m} &= m_{\max} - \frac{1}{\beta} \frac{\beta\left(m_{\max} - m_{\min}\right) - \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)}{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)} \\ &= \frac{m_{\min} - m_{\max}\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]} + \frac{1}{\beta}. \end{aligned} \tag{8}$$

The same result was given also by Hamilton (1968), in the form

$$\overline{m} = \frac{1}{\beta} + \frac{m_{\max}\exp\left[-\beta\, m_{\max}\right] - m_{\min}\exp\left[-\beta\, m_{\min}\right]}{\exp\left[-\beta\, m_{\max}\right] - \exp\left[-\beta\, m_{\min}\right]} \tag{9}$$

and Cosentino et al. (1976; 1977) as

$$\begin{aligned} \overline{m} &= \frac{1}{\beta} + m_{\max} + \frac{m_{\max} - m_{\min}}{\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] - 1} \quad \text{(ver. 1976)} \\ &= \frac{1}{\beta} + m_{\min} + \frac{m_{\max} - m_{\min}}{1 - \exp\left[\beta\left(m_{\max} - m_{\min}\right)\right]}. \quad \text{(ver. 1977)} \end{aligned} \tag{10}$$

All these three (or four) solutions (8)-(10) are equal and evaluated using moment method. Cosentino et al (1976, 1977) used this solution with variance to calculate the estimators for the maximum $m_{\max}$ and the β -value.

The equation (8) gives also the same estimator for β that was given by Page (1968)

$$\hat{\beta}_P = \left[\overline{m} - \frac{m_{\min} - m_{(N)}\exp\left(-\hat{\beta}_P\left(m_{(N)} - m_{\min}\right)\right)}{1 - \exp\left(-\hat{\beta}_P\left(m_{(N)} - m_{\min}\right)\right)}\right]^{-1} \tag{11}$$

where $m_{(N)}$ is a maximum observed value in catalogue C_N. Page derived his estimator using maximum likelihood method. We shall call this (11) estimator as a Page' estimator even the same is found by Hamilton and Cosentino et al. Also we can see from (8) that when $m_{\max} \to \infty$ then we have Aki-Utsu estimator $\beta_{AU}(\overline{m} - m_{\min}) = 1$. So this estimator (11) is related with the KS-2 function.

It was natural to wait that the maximum or extreme value model of the order statistic gives the same estimator than other models since the distribution functions are the same in case $n = 1$.

Owing to the relation between KS-1 and KS-2 given by (4), we can always easily change from the function to other. So there exists other variation of Page' estimator

$$\begin{aligned}\overline{m} &= m_{\max} - \frac{1}{\beta}\left\{\beta\left(m_{\max} - m_{\min}\right) - \frac{\beta\left(m_{\max} - m_{\min}\right) - \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)}{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)}\right\} \\ &= \frac{m_{\max} - m_{\min}\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]} - \frac{1}{\beta}.\end{aligned} \tag{12}$$

Thus

$$\hat{\beta}_P = \left[\frac{m_{(N)} - m_{\min}\exp\left(-\hat{\beta}_P\left(m_{(N)} - m_{\min}\right)\right)}{1 - \exp\left(-\hat{\beta}_P\left(m_{(N)} - m_{\min}\right)\right)} - \overline{m}\right]^{-1} \tag{13}$$

is other form of the Page' estimator and gives exactly same estimates than Page' original estimator. The estimator (12) does not exist as $m_{\max} \to \infty$ (it is valid only when $m_{\max} < \infty$) so this estimator (13) is related with the KS-1 function.

We showed before (Haarala and Orosco, 2016) that in unbounded case $(m_{\max} = \infty)$ the KS-1 is unbounded and the KS-2 gives

$$E\left(M_{(n)} \mid \infty\right) - m_{\min} = \frac{H_n}{\beta}, \tag{14}$$

where the $H_n = \sum_{k=1}^{n} k^{-1}$ is a harmonic number. Setting $n = 1$ and using the mean value estimator we have the Aki-Utsu estimator $\hat{\beta}_{AU}$ (Aki, 1965, and Utsu, 1965)

$$\hat{\beta}_{AU} = \frac{1}{\overline{m} - m_{\min}}.$$

Because in (14) H_n/β is a constant, it means that the distance between expected and minimum values does not depend how we choose the minimum. Always the distance from the arbitrary minimum to the expected value $E\left(M_{(n)} \mid \infty\right)$ is the same, when $m_{\max} = \infty$. This explain the bias what we can see in the Aki-Utsu estimator as $m_{\min} \to m_{\max}$. The mean $\overline{m}$ does not follow the minimum similar way than the expected value $E\left(M_{(n)} \mid \infty\right)$.

When the data comes from bounded system, then it is $\hat{E}\left(M_1 \mid m_{\max} < \infty\right) = \overline{m}$. Using KS-2 function to estimator we get

$$\frac{1}{\hat{\beta}_{AU}} = \overline{m} - m_{\min} = \frac{1}{\hat{\beta}} f_1^{KS-2}\left(\hat{\beta}\left(m_{\max} - m_{\min}\right)\right).$$

The relation

$$\hat{\beta} = \hat{\beta}_{AU} f_1^{KS-2}\left(\hat{\beta}\left(m_{\max} - m_{\min}\right)\right) \leq \hat{\beta}_{AU} \tag{15}$$

shows that Aki-Utsu estimator overestimate since $\hat{\beta} \leq \hat{\beta}_{AU}$ (equality holds only when $m_{\max} = \infty$). This gives also the correction function for the Aki-Utsu estimator

$$\hat{\beta} \approx \hat{\beta}_{AU}\, f_1^{KS-2}\left(\hat{\beta}_{AU}\left(\hat{m}_{\max} - m_{\min}\right)\right),$$

where the estimator $\hat{m}_{\max}$ is some estimator for $m_{\max}$.

Similarly, from (15) we can see that the Page's estimator and Aki-Utsu estimator has a relation

$$\hat{\beta}_P = \hat{\beta}_{AU} f_1^{KS-2}\left(\hat{\beta}_P\left(m_{(N)} - m_{\min}\right)\right).$$

Using the estimator $\overline{m}_{(n)}$ for $E\left(M_{(n)} \mid \infty\right)$, where $1 \leq n \leq N$, we have a generalized Aki-Utsu estimator

$$\hat{\beta}_{GAU}^{(n)} = \frac{H_n}{\overline{m}_{(n)} - m_{\min}}.$$

In the case $n = 1$ holds $\hat{\beta}_{AU} = \hat{\beta}_{GAU}^{(1)}$.

The Page' estimator

Similar way like the generalized Aki-Utsu estimator the Page' estimator can be solved by using

$$g_1\left(\tilde{\beta}_{GP}^{(n)}\right) = -\tilde{\beta}_{GP}^{(n)}\left(m_{\max} - \overline{m}_{(n)}\right) + f_n^{KS-1}\left(\tilde{\beta}_{GP}^{(n)}\left(m_{\max} - m_{\min}\right)\right)$$

or

$$g_2\left(\tilde{\beta}_{GP}^{(n)}\right) = \tilde{\beta}_{GP}^{(n)}\left(\overline{m}_{(n)} - m_{\min}\right) - f_n^{KS-2}\left(\tilde{\beta}_{GP}^{(n)}\left(m_{\max} - m_{\min}\right)\right).$$

Actually these functions are exactly the same since $g_2\left(\tilde{\beta}_{GP}^{(n)}\right) - g_1\left(\tilde{\beta}_{GP}^{(n)}\right) = 0$. We call this a generalized Page' estimator $\tilde{\beta}_{GP}^{(n)}$. In the case $n = 1$ holds $\tilde{\beta}_P = \tilde{\beta}_{GP}^{(1)}$.

In Figure 1 we show an example curve of an auxiliary function g with parameters $b = 1$, $m_{\max} = 8$, $m_{\min} = 5$ and $n = 40$. The function g has a parabolic form, so it can solved with Ridders' or Newton-Raphson method (Press et al., 1992).

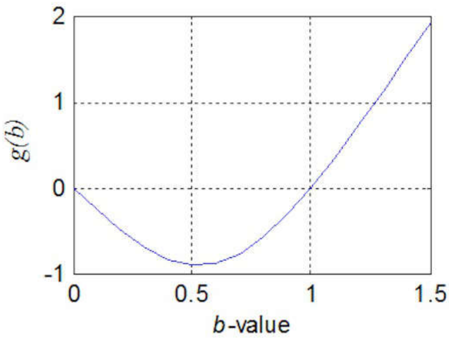

Figure 1: Example of the auxiliary function g

When making simulations (Haarala and Orosco, 2016) we showed how numerous artificial catalogues failed because of the condition $E\left(M_{(n)} \mid m_{\max}\right) \leq m_{\min} + H_n/\beta$. When we handle real seismic catalogues it is first calculated the estimator $\hat{\beta}$ and then with this estimator it is evaluated the estimator for the maximum. Due to the KS-2 function gives unique solution of the $\hat{\beta}^{(n)}$-value for each maximum $m_{\max}$, it can be said that $\hat{\beta}^{(n)}$ is a function of $m_{\max}$ (see Appendix A)

$$\hat{\beta}^{(n)}\left(m_{\max}\right)\left(\bar{m}_{(n)} - m_{\min}\right) = f_n^{KS-2}\left(\hat{\beta}^{(n)}\left(m_{\max}\right)\left(m_{\max} - m_{\min}\right)\right).$$

So the Page' estimator is $\hat{\beta}_P = \hat{\beta}^{(1)}\left(m_{(N)}\right)$ (where $m_{(N)}$ is the maximum observed value of catalogue) and the Aki-Utsu estimator $\hat{\beta}_{AU} = \hat{\beta}^{(1)}(\infty)$. These estimators hold the inequality

$$\hat{\beta}_{GP}^{(n)} = \hat{\beta}^{(n)}\left(m_{(n)}\right) \leq \hat{\beta}^{(n)}\left(\hat{m}_{\max}\right) \leq \hat{\beta}^{(n)}(\infty) = \hat{\beta}_{GAU}^{(n)}.$$

This shows that the β-value measures the distance between the minimum and the infinity similar way than the expected value of maximum

$$E\left(M_{(n)} \mid m_{(n)}\right) \leq E\left(M_{(n)} \mid \hat{m}_{\max}\right) \leq E\left(M_{(n)} \mid \infty\right).$$

It is well known that when $\beta \to \hat{\beta}_{AU}$ then $m_{\max} \to \infty$. (Normally this is presented in an opposite way as $m_{\max} \to \infty$ then $\hat{\beta}_{AU} \to \beta$.) A lesser known result is a limit when $\beta \to 0$. Let consider the KS-2 function

$$E\left(M_{(n)} \mid m_{\max}\right) - m_{\min} = \frac{n}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^k}{k(k+n)}. \quad (16)$$

Because of

$$1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] = \beta\left(m_{\max} - m_{\min}\right)\left\{1 + \sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{\max} - m_{\min}\right)\right]^{j-1}}{j!}\right\}$$

the equation (16) results

$$E\left(M_{(n)}\mid m_{\max}\right)-m_{\min}=\frac{n}{n+1}\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{j-1}}{j!}\right\}$$

$$+n\sum_{k=2}^{\infty}\frac{\beta^{k-1}\left(\left(m_{\max}-m_{\min}\right)\left\{1+\sum_{j=2}^{\infty}\frac{\left[-\beta\left(m_{\max}-m_{\min}\right)\right]^{j-1}}{j!}\right\}\right)^{k}}{k\left(k+n\right)}.$$

If $\beta\to 0$ then we have

$$m_{\max}=m_{\min}+\frac{n+1}{n}\left[E\left(M_{(n)}\mid m_{\max}\right)-m_{\min}\right]. \tag{17}$$

For example, in the case $n=2$ this gives $m_{\max}\geq m_{(1)}+3/2\left(m_{(2)}-m_{(1)}\right)=m_{(2)}+\left(m_{(2)}-m_{(1)}\right)/2$. Since it is a limit as $\beta\to 0$ it does not mean that the maximum is just two times the distance between minimum and mean. Actually is a minimum distance to the maximum i.e.

$$m_{\max}\geq m_{\min}+\frac{n+1}{n}\left(\overline{m}_{(n)}-m_{\min}\right)$$

Up to this point we can apply these concepts in the case we have a seismic catalogue where we take the two biggest magnitudes, say for example 8.8 and 9.5, so this is our sub-catalogue of size $n=2$. We have then $\hat{m}_{\max}\geq 9.5+(9.5-8.8)/2=9.85$. This is a statistical estimator so even though the formula holds that the maximum is equal or bigger than the result it gives, it could be smaller. The advantage of this approach is that we do not need to know the β-value to have some estimation of the maximum and besides, it is easy to calculate.

We could also find the minimum convergence magnitude to the Page' estimator. The equation (17) states now that there is solution only if

$$\overline{m}_{(n)}<m_{\min}+\frac{n}{n+1}\left(m_{\max}-m_{\min}\right). \tag{18}$$

where $m_{\max}$ can be replaced with its estimator.

Actually there is no numerical solution when the equality holds i.e. when $\beta=0$. That is to say that if the estimator of the expected value $\overline{m}_{(n)}$ is equal or bigger than this limit then the estimator $\hat{\beta}_{GP}^{(n)}$ has no solution. In that case the $\hat{\beta}$ value will be zero or negative when the inequality fails so we can define $\hat{\beta}_{GP}^{(n)}=0$ in those cases.

In figure 2 we plotted the Page' estimator $\hat{\beta}_{GP}^{(n)}$ and the Aki-Utsu estimator $\hat{\beta}_{GAU}^{(n)}$ for the case: $b=1$, $m_{\max}=8$, $m_{\min}=5$ and $n=200$. There is only one artificial catalogue, but the estimators calculated considering $1<n<N$ have been «smoothed» taking 1000 random sets of sub-catalogues and calculating the mean for them. We can see that basically $\left[\hat{\beta}_{GP}^{(n)},\hat{\beta}_{GAU}^{(n)}\right]$ grows when the size of sub-catalogue grows. In the very first sub-catalogues the correct β is not included into the interval. The figure 2 has been made with Ridders' method (Press et al, 1992) starting with the interval $\left[\log(10)/10^{8},\hat{\beta}_{GAU}^{(n)}+\log(10)/10\right]$.

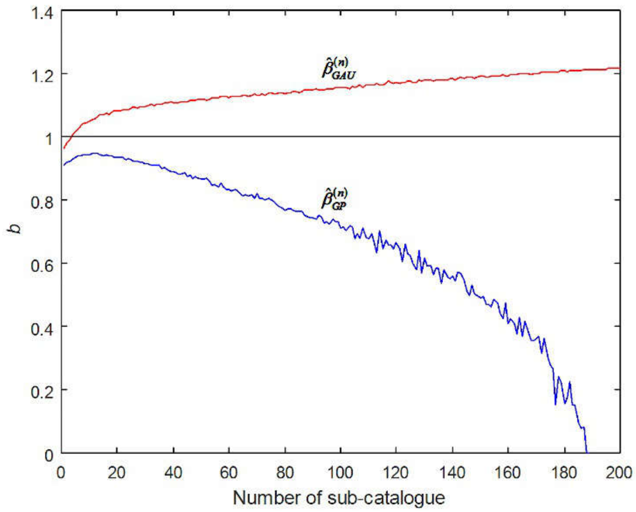

Figure 2: Page' and Aki-Utsu estimators with different sub-catalogue sizes using Ridders' method.

The exact variance

Comparing to the calculus of the first moment (Haarala and Orosco, 2016), the calculus of the variance is much more complicated. Integrating by parts it is

$$E\left(M_{(n)}^2 \mid m_{\max}\right)=\int_{m_{\min}}^{m_{\max}} m^2\, dF_{M_{(n)}}\left(m \mid m_{\max}\right)=m_{\max}^2-2\int_{m_{\min}}^{m_{\max}} m\, F_{M_{(n)}}\left(m \mid m_{\max}\right) dm. \tag{19}$$

When we applied the functions (1) and (2) the integral can be written

$$\int_{m_{\min}}^{m_{\max}} m\, F_{M_{(n)}}\left(m \mid m_{\max}\right) dm=\frac{\int_{m_{\min}}^{m_{\max}} m\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n dm}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n}.$$

Because of the integrate function of $\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n$ is

$$\int\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n dm=\left[1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right]^n \frac{1}{\beta} f_n^{KS-1}\left(\beta\left(m-m_{\min}\right)\right) \tag{20}$$

(the proof has been given in the Appendix B), integrating again by parts the second moment (19) yields to

$$E\left(M_{(n)}^2 \mid m_{\max}\right) = m_{\max}^2 - \frac{2m_{\max}}{\beta} f_n^{KS-1}\left(\beta\left(m_{\max} - m_{\min}\right)\right)$$
$$+\frac{2}{\beta}\int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^n f_n^{KS-1}\left(\beta\left(m-m_{\min}\right)\right) dm. \tag{21}$$

Using notation $\Delta_n\left(\mathfrak{m}\right) = \beta^{-1} f_n^{KS-1}\left(\beta\left(\mathfrak{m} - m_{\min}\right)\right)$ the second moment is

$$E\left(M_{(n)}^2 \mid m_{\max}\right) = m_{\max}^2 - 2m_{\max}\Delta_n\left(m_{\max}\right) + 2\int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^n \Delta_n\left(m\right) dm.$$

Hence

$$\left[E\left(M_{(n)} \mid m_{\max}\right)\right]^2 = \left[m_{\max} - \Delta_n\left(m_{\max}\right)\right]^2$$
$$= m_{\max}^2 - 2m_{\max}\Delta_n\left(m_{\max}\right) + \Delta_n^2\left(m_{\max}\right)$$

so the variance can be written

$$Var\left(M_{(n)} \mid m_{\max}\right) = E\left(M_{(n)}^2 \mid m_{\max}\right) - \left[E\left(M_{(n)} \mid m_{\max}\right)\right]^2$$
$$= 2\int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^n \Delta_n\left(m\right) dm - \Delta_n^2\left(m_{\max}\right). \tag{22}$$

Since for all m holds $\Delta_n\left(m\right) \le \Delta_n\left(m_{\max}\right)$ the variance has an upper limit

$$Var\left(M_{(n)} \mid m_{\max}\right) \le 2\Delta_n\left(m_{\max}\right)\int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^n dm - \Delta_n^2\left(m_{\max}\right) = \Delta_n^2\left(m_{\max}\right).$$

Next we need to solve the integral in the equation (21). Assuming that n is integer the KS-1 function can be written using finite sum as (see Haarala and Orosco, 2016)

$$f_n^{KS-1}\left(\beta\left(m-m_{\min}\right)\right) = \frac{\beta\left(m-m_{\min}\right) - \sum_{k=1}^{n} \frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^k}{k}}{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^n}.$$

Now the integral in (21) can be written

$$\frac{2}{\beta}\int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^n f_n^{KS-1}\left(\beta\left(m-m_{\min}\right)\right) dm =$$

$$= \frac{\dfrac{2}{\beta}\displaystyle\int_{m_{\min}}^{m_{\max}} \beta\left(m-m_{\min}\right)-\sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k}}{k}\,dm}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}}$$

$$= \frac{\left(m_{\max}-m_{\min}\right)^{2}-\dfrac{2}{\beta}\displaystyle\sum_{k=1}^{n}\left\{\frac{1}{k}\int_{m_{\min}}^{m_{\max}}\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k}dm\right\}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}}$$

The integral in the sum is given in (20) . We can write the exact variance as

$$Var\left(M_{(n)}\mid m_{\max}\right)=\frac{\beta^{2}\left(m_{\max}-m_{\min}\right)^{2}-2\displaystyle\sum_{k=1}^{n}\left\{\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k}f_{k}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right\}}{\beta^{2}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}}-\Delta_{n}^{2}\left(m_{\max}\right)$$
$$=\frac{\Delta_{0}^{2}\left(m_{\max}\right)-\dfrac{2}{\beta}\displaystyle\sum_{k=1}^{n}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}\Delta_{k}\left(m_{\max}\right)}{k}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}}-\Delta_{n}^{2}\left(m_{\max}\right). \qquad (23)$$

Without applying series this variance looks like (the proof is given in the Appendix C)

$$Var\left(M_{(n)}\mid m_{\max}\right)=\frac{\left[\beta\left(m_{\max}-m_{\min}\right)-H_{n}\right]^{2}-H_{n}^{2}+2\displaystyle\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k\left(k+j\right)}}{\beta^{2}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}}$$
$$-\left\{\frac{\beta\left(m_{\max}-m_{\min}\right)-\displaystyle\sum_{k=1}^{n}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k}}{\beta\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{n}}\right\}^{2}. \qquad (24)$$

Both expressions of the variance (equations(23) and (24) are numerically unstable. The numerically stable form of variance can be written as (proof in Appendix D)

$$Var\left(M_{(n)}\mid m_{\max}\right)=\frac{1}{\beta^{2}}\sum_{k=2}^{\infty}\frac{2n}{2n+k}\left\{\sum_{j=1}^{k-1}\frac{1}{n+j}\right\}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{n+k}. \qquad (25)$$

The series can be solved similar way we made with the KS-1 and KS-2 (Haarala and Orosco, 2016). Moreover, the expression (25) has continuous variable n . That means that we could replace n with $T\lambda$, where T is a continuous variable of time and λ is a rate of events in some time unit (normally it is a year).

The variance(25) can be calculated by means of our MATLAB function (Haarala and Orosco, 2016) as

$$Var\left(M_{(n)} \mid m_{\max}\right) = \frac{1}{\beta^2} KS(3, \beta\left(m_{\max} - m_{\min}\right), n).$$

In Figure 3 we show an example of the calculus of the variance for the case when $b = 1$, $m_{\max} = 8$ and $m_{\min} = 5$ given different sizes of catalogues. The variance has the shape of the Gamma function. The maximum value is at $n = 7$; from size $n = 65$ on, the level of variance is less than the level of variance when the catalogue size is $n = 1$.

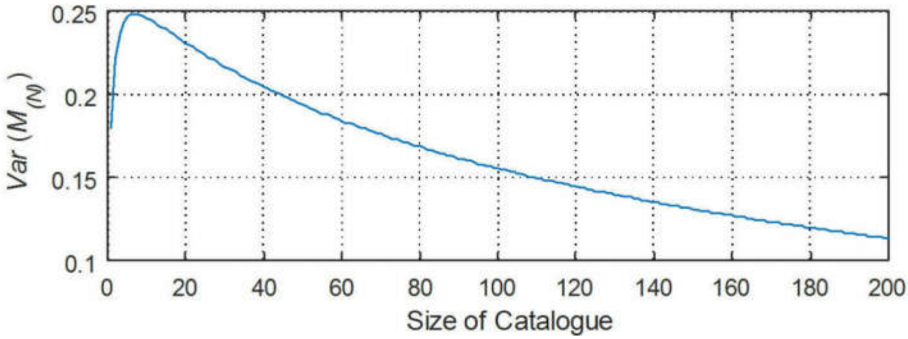

Figure 3: Example curve of the variance.

More about the variance

As we have shown, the first moment is related with the maximum likelihood estimators (Aki, 1965; Page, 1968) and other moment estimators (Hamilton, 1967; Cosentino et al, 1977), so it is quite expected that the exact variance will be the same in the case $n = 1$.

In the formula (24) we set $n = 1$, then we have

$$Var\left(M_{(1)} \mid m_{\max}\right) = \frac{1}{\beta^2} - \left[\frac{m_{\max} - m_{\min}}{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right]^2 \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right] \tag{26}$$

We can find the variance given by Aki (1965) as $m_{\max} \to \infty$:

$$Var\left(M_{(1)} \mid \infty\right) = \frac{1}{\beta^2}.$$

Generalizing, the variance at $m_{\max} = \infty$ is

$$Var\left(M_{(n)} \mid \infty\right) = \frac{1}{\beta^2} \sum_{k=1}^{n} \frac{1}{k^2}.$$

We can get it starting from (25). The proof is given in the Appendix E. It shows that the variance is a bounded function for all m_{max} (and also for all n) and it is smaller than $1.65/\beta^2$.

The variance (26) is equal to the variance as it was given by Hamilton (1967)

$$Var\left(M_{(1)} \mid m_{max}\right) = \frac{1}{\beta^2} - \left[\frac{m_{max} - m_{min}}{\exp\left[\frac{\beta(m_{max} - m_{min})}{2}\right] - \exp\left[-\frac{\beta(m_{max} - m_{min})}{2}\right]}\right]^2$$

(actually Hamilton wrote $\exp[z] - \exp[-z] = 2\sinh[z]$) and the variance was given by Cosentino et al (1976, 1977)

$$\begin{aligned} Var\left(M_{(1)} \mid m_{max}\right) &= \frac{1}{\beta^2} - \left[m_{max} + \frac{1}{\beta} - \bar{m}\right]^2 \exp\left[-\beta(m_{max} - m_{min})\right] \quad \text{(ver. 1976)} \\ &= \frac{1}{\beta^2} - \left[m_{min} + \frac{1}{\beta} - \bar{m}\right]^2 \exp\left[\beta(m_{max} - m_{min})\right] \quad \text{(ver. 1977)} \end{aligned}$$

where they used the equation (10) to replace the square of brackets in (26).

We have showed now how variance fits with the models of Hamilton (1967) and Cosentino et al (1976; 1977). It is natural since the distribution function is the same in the case $n = 1$. If we write the variance (26) as

$$Var\left(M_{(1)} \mid m_{max}\right) = \frac{\partial}{\partial\beta}\left\{-\frac{1}{\beta} + \frac{(m_{max} - m_{min})}{1 - \exp\left[-\beta(m_{max} - m_{min})\right]} + C\right\}$$

where C is some constant, for example $C = \bar{m} - m_{max}$. Using the equation of the variance of Cosentino et al (1976) (10) , we have some function g such that

$$g(\beta) = \bar{m} - m_{max} - \frac{1}{\beta} + \frac{m_{max} - m_{min}}{1 - \exp\left[-\beta(m_{max} - m_{min})\right]}.$$

Now $g(\beta) = 0$ because of the equation (10), and $g'(\beta) = Var\left(M_{(1)} \mid m_{max}\right)$. So the g is maximum likelihood function L

$$\begin{aligned} g(\beta) &= \frac{1}{N}\frac{\partial}{\partial\beta} L(\beta; m) \\ &= \frac{1}{N}\frac{\partial}{\partial\beta}\left\{\log \prod_{k=1}^{N} \frac{\beta \exp\left[-\beta(m_k - m_{min})\right]}{1 - \exp\left[-\beta(m_{max} - m_{min})\right]}\right\}. \end{aligned} \qquad (27)$$

Simulations

In figure 4 we display the mean estimator for the expected value of $M_{(N)}$ given by (6) . We generated only one random catalogue C_{200} with parameters $b=1$, $m_{\max}=8$, $m_{\min}=5$ and $n=200$ (figure 4a); using this catalogue we take randomly the events from the catalogue C_{200} into the sub-catalogues $C_{n:k}$. The means $\overline{m}_{(1)}$ and $\overline{m}_{(200)}$ are fixed. For the sub-catalogue of sizes $1<n<200$ we get different paths. The continuous thick line represents the mean value of 1000 samples and the dashed line is the theoretical expected value. We can see that the mean values are quite close to the expected values in the range of small catalogue sizes. Actually this is quite expected since for example for the n-values 1, 2 and 3 the number of sub-catalogues are 200, 100 and 66, respectively, and the estimator of the expected value is a mean value of the maximums of those sub-catalogues. The figure 4b is the same than the 4a but it is generated with parameters $b=1$, $m_{\max}=8$, $m_{\min}=6$ and $n=200$.

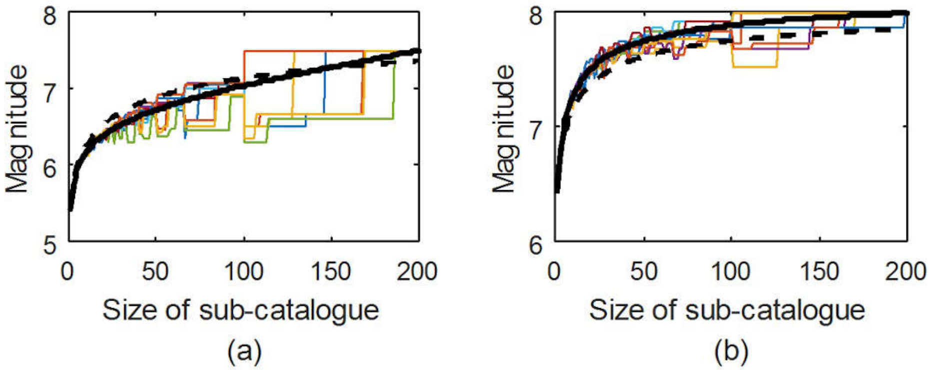

Figure 4: The estimator (6) for some catalogue C_{200} with different parameters

We generated 10000 catalogues with parameters $b=1$, $m_{\max}=8$, $m_{\min}=5$ and $n=200$. For those catalogues we have calculated the mean $\overline{m}_{(n)}$. In the figures 5a, 5b, 5c and 5d we show the histograms of means $\overline{m}_{(1)}$, $\overline{m}_{(10)}$, $\overline{m}_{(100)}$ and $\overline{m}_{(200)}$ respectively.

Also in the figures is pointed out a limit $E\left(M_{(n)} \mid \infty\right)$ when we set $b=1$ (white line). We can see how close this limit is to the expected value, which is estimated with the mean value. We can also see that the distribution function is

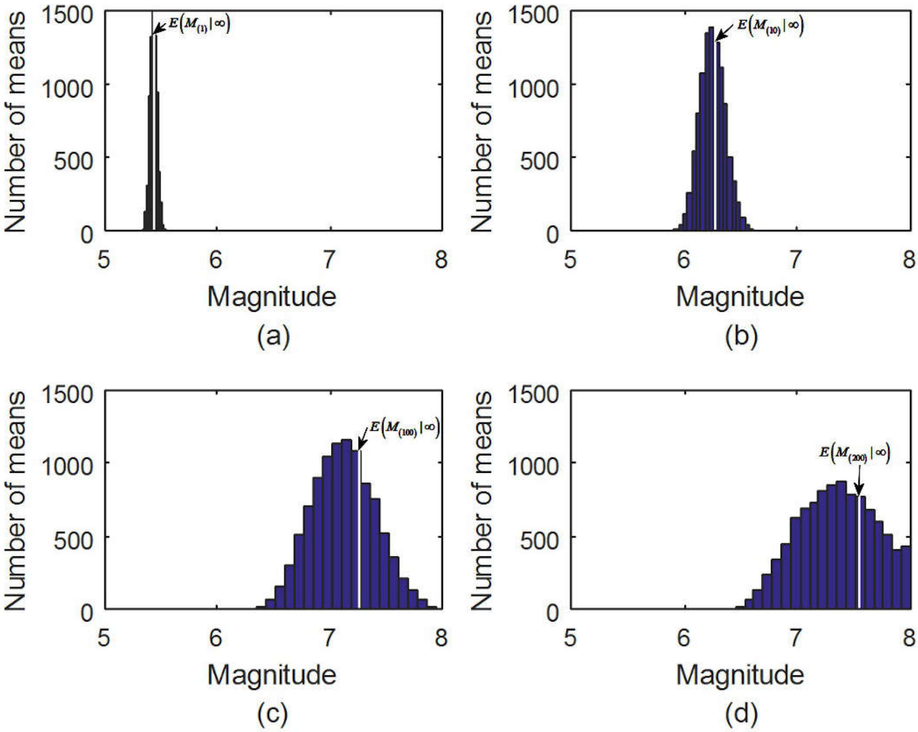

Figure 5: Distribution of mean values

normal (as it should be because of the central limit theorem) except in the figure 5d. Because the probability that the event is outside of interval $[m_{\min}, m_{\max}]$ is zero, the probability distribution function is actually a double truncated normal distribution function. Of course the normalization factor is about one for all cases except in the case 5d, so we can use the unbounded normal distribution function in those cases. Moreover, we can see that the variance is much smaller in case 5a than cases 5b-5d.

In the figure 6a we plotted a mean of the estimators $\hat{m}_{\max}$ using the same β -value (as we made in the Part I) that we used to generate the artificial catalogues, and the Page' estimator $\hat{\beta}_P$ (i.e. Page' estimator is calculated at $n=1$ using maximum observed value $m_{(N)}$). We have got Figure 6a by generating 1000 artificial catalogues with parameters $b=1$, $m_{\max}=8$, $m_{\min}=5$ and $1 \leq n \leq 200$. The Figure 6b shows how many catalogues could be used to calculate these mean values.

Because of the Page' estimator underestimates the β -value we can use more catalogues in the calculus but still 25-30% catalogues are rejected, which provokes the bias to the mean value of the maximums.

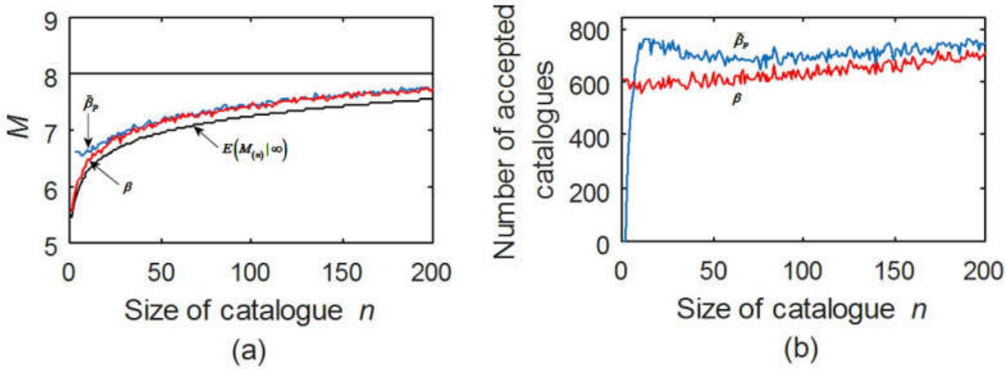

Figure 6: Estimation of the maximum $\hat{m}_{\max}$

Concluding remarks

In this work we showed that the Aki and Page's (Maximum Likelihood) estimators and Hamilton, Cosentino et al. and Utsu's (Moment) estimators, can be evaluated by using KS functions. When the sub-catalogue size is $n = 1$, all the estimators gave the same values when we use the same parameters. The differences, if any, come from the computation.

In order statistic we can generalize these results to the family of estimators. Due to the method we are here proposing joins with moment estimator method when $n = 1$, we could solve the estimators β and $m_{\max}$ using first and second moments, even though we do not use this approach in this series of works, but it was made in earlier ones. Any way we showed that the order statistic carries more information than Moment Estimator Method, because we could apply the Moment Estimator Method for each $1 \le n \le N$.

Appendix A

We consider the function

$$y = f_n^{KS-2}(x) = n\sum_{k=1}^{\infty}\frac{\left(1-\exp[-x]\right)^k}{k(k+n)}. \tag{28}$$

It has a derivative

$$\begin{aligned}\left(f_n^{KS-2}\right)'(x) &= n\exp[-x]\sum_{k=1}^{\infty}\frac{\left(1-\exp[-x]\right)^{k-1}}{(k+n)}\\ &= n\exp[-x]\left[\frac{1}{(1+n)}+f_{n+1}^{KS-1}(x)\right].\end{aligned}$$

We can see that $\left(f_n^{KS-2}\right)'(x) > 0$ for all $x \in [0,\infty[$, so the function $f_n^{KS-2}(x)$ is a strictly increasing function. This means that it is a bijection from $[0,\infty[$ onto $[0,H_n[$ and it has a unique solution.

Let's consider now the equation

$$\beta\left(\overline{m}_{(n)} - m_{\min}\right) = f_n^{KS-2}\left(\beta\left(m_{\max} - m_{\min}\right)\right). \tag{29}$$

Assume that there are some $\hat{\beta}_1^{(n)}, \hat{\beta}_2^{(n)}$ and $m_{\max,1}, m_{\max,2}$, respectively. If

$$f_n^{KS-2}\left(\hat{\beta}_1^{(n)}\left(m_{\max,1} - m_{\min}\right)\right) = f_n^{KS-2}\left(\hat{\beta}_2^{(n)}\left(m_{\max,2} - m_{\min}\right)\right)$$

then $\hat{\beta}_1^{(n)}\left(\overline{m}_{(n)} - m_{\min}\right) = \hat{\beta}_2^{(n)}\left(\overline{m}_{(n)} - m_{\min}\right)$ and $\hat{\beta}_1^{(n)} = \hat{\beta}_2^{(n)}$. Moreover, since $f_n^{KS-2}(x)$ is the bijection, then $\hat{\beta}_1^{(n)}\left(m_{\max,1} - m_{\min}\right) = \hat{\beta}_2^{(n)}\left(m_{\max,2} - m_{\min}\right)$. But now $\hat{\beta}_1^{(n)} = \hat{\beta}_2^{(n)}$, so also $m_{\max,1} = m_{\max,2}$. That is to say, if $\hat{\beta}_1^{(n)}, \hat{\beta}_2^{(n)}$ and $m_{\max,1}, m_{\max,2}$ are from $y = f_n^{KS-2}(x)$ then $m_{\max,1} = m_{\max,2}$ and $\hat{\beta}_1^{(n)} = \hat{\beta}_2^{(n)}$. This makes the one-to-one mapping between $\hat{\beta}^{(n)}$ and $m_{\max}$.

For each $\beta = \hat{\beta}^{(n)} \in \left]0, \hat{\beta}_{AU}^{(n)}\right[$ there is an image $m_{\max} \in \left]m_{\min} + (n+1)\left(\overline{m}_{(n)} - m_{\min}\right)/n, \infty\right[$ and for each we can find. This follows the reality that $\beta \to \hat{\beta}_{AU}^{(n)}$ when $m_{\max} \to \infty$, and $m_{\max} \to m_{\min} + (n+1)\left(\overline{m}_{(n)} - m_{\min}\right)/n$ when $\beta \to 0$. Thus the equation (29) defines the bijection mapping between the sets $\left]0, \hat{\beta}_{AU}^{(n)}\right[$ and $\left]m_{\min} + (n+1)\left(\overline{m}_{(n)} - m_{\min}\right)/n, \infty\right[$.

Appendix B

Firstly, it is to note that the derivative of the KS-2 function is

$$\begin{aligned}\frac{\partial}{\partial\mathfrak{M}} f_n^{KS-2}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right) &= n\sum_{k=1}^{\infty}\frac{\partial}{\partial\mathfrak{M}}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^k}{k\left(k+n\right)} \\ &= n\beta\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\sum_{k=1}^{\infty}\frac{k\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^{k-1}}{k\left(k+n\right)} \\ &= n\beta\frac{\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]} f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right).\end{aligned}$$

Thus the derivative of the KS-1 function is

$$\begin{aligned}\frac{1}{\beta}\frac{\partial}{\partial\mathfrak{M}} f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right) &= \frac{\partial}{\partial\mathfrak{M}}\left[\left(\mathfrak{M}-m_{\min}\right)-\frac{1}{\beta}f_n^{KS-2}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right)\right] \\ &= 1-n\frac{\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]} f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right).\end{aligned}$$

So the direct derivative gives

$$\begin{aligned}&\frac{\partial}{\partial\mathfrak{M}}\left[\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^n\frac{1}{\beta}f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right)\right] \\ &= n\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^{n-1} f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right) \\ &\quad+\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^n\left\{1-n\frac{\exp\left(-\beta\left(\mathfrak{M}-m_{\min}\right)\right)}{1-\exp\left(-\beta\left(\mathfrak{M}-m_{\min}\right)\right)}f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right)\right\} \\ &= \left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^n,\end{aligned}$$

and we have

$$\int\left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^n d\mathfrak{M} = \left(1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]\right)^n\frac{1}{\beta}f_n^{KS-1}\left(\beta\left(\mathfrak{M}-m_{\min}\right)\right)+C. \tag{30}$$

Appendix C

Because of

$$\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k}=\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k-1} -\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k-1}\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]$$

so the integral

$$\begin{aligned}&\int_{m_{\min}}^{m_{\max}}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k}}{k+j}d\mathfrak{m}\\ &=\int_{m_{\min}}^{m_{\max}}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k-1}}{k+j}d\mathfrak{m}-\frac{1}{\beta}\sum_{k=1}^{n-j}\int_{m_{\min}}^{m_{\max}}\frac{\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k-1}\beta\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}{k+j}d\mathfrak{m}\\ &=\frac{m_{\max}-m_{\min}}{j+1}+\int_{m_{\min}}^{m_{\max}}\sum_{k=1}^{n-(j+1)}\frac{\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k}}{k+(j+1)}d\mathfrak{m}-\frac{1}{\beta}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k(k+j)},\end{aligned}$$

where $j=0,1,\ldots,n-1$, is applied n times; then it gives for the integral

$$\begin{aligned}&\frac{2}{\beta}\int_{m_{\min}}^{m_{\max}}\sum_{k=1}^{n}\frac{\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{k}}{k}d\mathfrak{m}\\ &=\frac{2}{\beta}\left(m_{\max}-m_{\min}\right)H_n-\frac{2}{\beta^2}\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k(k+j)}.\end{aligned}$$

Now the variance gets the form

$$\begin{aligned}Var\left(M_{(n)}\mid m_{\max}\right)&=\frac{\beta^2\left(m_{\max}-m_{\min}\right)^2-2\beta\left(m_{\max}-m_{\min}\right)H_n+2\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k(k+j)}}{\beta^2\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n}-\Delta_n^2\left(m_{\max}\right)\\ &=\frac{\left[\beta\left(m_{\max}-m_{\min}\right)-H_n\right]^2+2\sum_{j=0}^{n-1}\sum_{k=1}^{n-j}\frac{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{k}}{k(k+j)}-H_n^2}{\beta^2\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^n}-\Delta_n^2\left(m_{\max}\right).\end{aligned}$$

Appendix D

To get the numerically stable form for the variance, we start from (22)

$$Var\left(M_{(n)} \mid m_{\max}\right) = \frac{2}{\beta}\int_{m_{\min}}^{m_{\max}}\left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^{n} f_{n}^{KS-1}\left(\beta\left(m-m_{\min}\right)\right)dm - \left[\frac{1}{\beta} f_{n}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right]^{2}.$$

The integral can be written as

$$\begin{aligned}
&\frac{2}{\beta}\int_{m_{\min}}^{m_{\max}}\left[\frac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^{n} f_{n}^{KS-1}\left(\beta\left(m-m_{\min}\right)\right)dm \\
&= \frac{\frac{2}{\beta}\int_{m_{\min}}^{m_{\max}} \beta\left(m-m_{\min}\right) - \sum_{k=1}^{n}\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k}}{k}dm}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}} \\
&= \frac{\frac{2}{\beta}\sum_{k=n+1}^{\infty}\left\{\frac{1}{k}\int_{m_{\min}}^{m_{\max}}\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k} dm\right\}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{n}} \\
&= \frac{2}{\beta}\sum_{k=n+1}^{\infty}\left\{\frac{1}{k}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k-n}\frac{1}{\beta} f_{k}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right\} \\
&= \frac{2}{\beta^{2}}\sum_{k=1}^{\infty}\left\{\frac{1}{k+n}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k} f_{k+n}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right\}
\end{aligned} \tag{31}$$

because

$$\beta\left(m-m_{\min}\right) = \sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k}}{k} = f_{0}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)$$

and using the integral formula (30) we get

$$f_{m}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) = \sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{j}}{j+m},$$

then the last series in the (31) can be written as

$$\frac{2}{\beta^2}\sum_{k=1}^{\infty}\left\{\frac{1}{k+n}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k f_{k+n}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right\}$$
$$=\frac{2}{\beta^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k+j}}{(k+n)(j+k+n)}. \tag{32}$$

Now it is to collect terms containing the same power so we have for some $k'=k+j$, $k'=2,3,4,\ldots$,

$$\sum_{j=1}^{k'-1}\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k'}}{(k'-j+n)(k'+n)}=\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k'}}{k'+n}\sum_{j=1}^{k'-1}\frac{1}{n+k'-j}$$
$$=\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k'}}{k'+n}\sum_{j'=1}^{k'-1}\frac{1}{n+j'}.$$

Applying this into (32) and rewriting the index of series we get

$$\frac{2}{\beta^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k+j}}{(k+n)(j+k+n)}=\frac{2}{\beta^2}\sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{1}{n+j}\right\}\frac{\left(1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right)^{k}}{n+k}, \tag{33}$$

so the variance is given by

$$Var\left(M_{(n)}\mid m_{\max}\right)=\frac{2}{\beta^2}\sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{1}{n+j}\right\}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{n+k}$$
$$-\left[\frac{1}{\beta}f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right]^2. \tag{34}$$

We could also write $\left[\beta^{-1}f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right]^2$ similarly as the series (32)

$$\frac{1}{\beta^2}\sum_{k=1}^{\infty}\left\{\frac{1}{k+n}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k f_{n}^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right\}$$
$$=\frac{1}{\beta^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k+j}}{(n+k)(n+j)}. \tag{35}$$

In the same way as we got (33) above, the (35) gives

$$\left[\frac{1}{\beta}f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right)\right]^2=\frac{1}{\beta^2}\sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{1}{(n+k-j)(n+j)}\right\}\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}. \tag{36}$$

Applying (36) into (34) , we get for variance the formula

$$Var\left(M_{(n)} \mid m_{\max}\right) = \frac{1}{\beta^2} \sum_{k=2}^{\infty} \left\{ \sum_{j=1}^{k-1} \frac{k+n-2j}{(n+j)(n+k-j)} \right\} \frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{n+k}.$$

We could use this form of variance, but the inner sum produces a small error (bias) for the results because it is a sum of positive and negative values. To avoid this problem, we can calculate the pairs $(1,k-1),(2,k-2),\ldots,(j,k-j)$. If k is even, then there is a $k/2$ term without pair. We have now

$$\frac{k+n-2j}{(n+j)(n+k-j)} + \frac{k+n-2(k-j)}{(n+(k-j))(n+k-(k-j))} = \frac{2n}{(n+j)(n+k-j)}$$

$$\frac{k+n-2(k/2)}{(n+(k/2))(n+k-(k/2))} = \frac{n}{(n+(k/2))(n+k-(k/2))}$$

so we can write

$$\sum_{j=1}^{k-1} \frac{k+n-2j}{(n+j)(n+k-j)} = \sum_{j=1}^{k-1} \frac{n}{(n+j)(n+k-j)}. \tag{37}$$

The factor (37) is not also so good because we need to accelerate it. In the worst case the van Wijngaarden transformation uses the value $k = 2^{57}$ which means that the factor (37) would need a huge capacity of computation. The factor in the equation (34) is better, since it can accelerate its factor as

$$\sum_{j=1}^{k} \frac{1}{j+n} = \sum_{m=1}^{n+k} \frac{1}{m} - \sum_{m=1}^{n} \frac{1}{m} = H_{n+k} - H_n$$

where the harmonic number H_n can be calculated for example using Ramanujan's approximation (Villarino, 2008)

$$\begin{aligned} m &= \frac{n(n+1)}{2} \\ H_n &\approx \frac{1}{2}\log(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} \\ &\quad - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \frac{191}{360360m^6} \end{aligned} \tag{38}$$

for $n \geq 10$.

MATLAB has a *harmonic* function which bases to the Psi function (Appendix E). It is much slower than the Ramanujan's approximation. Only problem with the Ramanujan's approximation is that it is not so good for values less than 10. There is no sense to add more terms to the sum, because it grows the time of calculus. So when we have some value $n < 10,\ n \in \mathbb{R}$, then an easy trick is to find an integer D such that $n' = n + D \in [10,11[$ when we have $H_n = H_{n'} - \sum_{j=1}^{D} 1/(n+j)$.

In the Figure 7 we show the absolutely error between the Ramanujan's approximation and the Psi function calculating subtraction $d = H_n - H_{0.5}$, $n = 1.5, 2.5, 3.5, \ldots, 9999.5$ with both method and calculating the difference of these results. Even the d is calculated with different methods, the difference between the results of the methods is less than 6×10^{-15}.

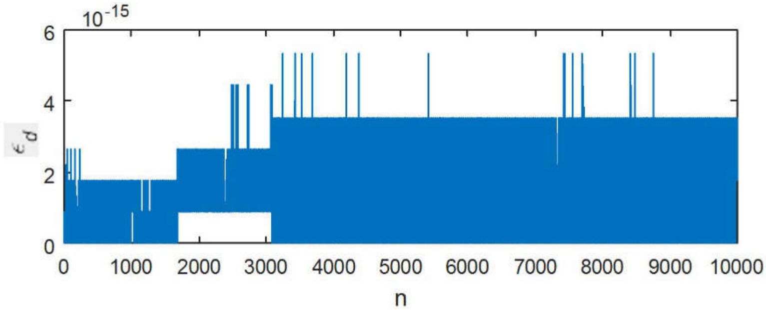

Figure 7: Error between Psi function method and Ramanujan's approximation method

To accelerate the factor (37) we need to modify it. It is clear that

$$\frac{1}{(n+j)(n+k-j)} = \frac{1}{2n+k}\left[\frac{1}{(n+j)} + \frac{1}{(n+k-j)}\right].$$

We get now

$$\begin{aligned}\sum_{j=1}^{k-1} \frac{n}{(n+j)(n+k-j)} &= \frac{n}{2n+k}\left[\sum_{j=1}^{k-1}\frac{1}{n+j} + \sum_{j=1}^{k-1}\frac{1}{n+k-j}\right] \\ &= \frac{2n}{2n+k}\sum_{j=1}^{k-1}\frac{1}{n+j}.\end{aligned}$$

The final variance can be written as

$$Var\left(M_{(n)} \mid m_{\max}\right) = \frac{1}{\beta^2}\sum_{k=2}^{\infty}\frac{2n}{2n+k}\left\{\sum_{j=1}^{k-1}\frac{1}{n+j}\right\}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^k}{n+k}.$$

This is a positive term series (numerical stable) and we can accelerate it with the Ramanujan's approximation (38).

Appendix E

Let the variance to be at $m_{\max} = \infty$

$$\beta^2 Var\left(M_{(n)} \mid m_{\max} = \infty\right) = \sum_{k=2}^{\infty} \frac{2n}{(2n+k)(n+k)} \left\{ \sum_{j=1}^{k-1} \frac{1}{n+j} \right\}. \tag{39}$$

Because (39) is a series of nonnegative terms so it converges absolutely and then every rearrangement of the series converges (39), and they all converge to the same sum (Rudin, 1987). We see now that the terms of the series are

$$\begin{array}{ll} \frac{2n}{(2n+2)(n+2)}\left\{\frac{1}{n+1}\right\}, & k=2 \\ \frac{2n}{(2n+3)(n+3)}\left\{\frac{1}{n+1}+\frac{1}{n+2}\right\}, & k=3 \\ \frac{2n}{(2n+4)(n+4)}\left\{\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right\}, & k=4 \\ \vdots & \end{array}$$

so the rearrangement of the series (39) gives

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{2n}{(2n+k)(n+k)} \left\{ \sum_{j=1}^{k-1} \frac{1}{n+j} \right\} &= \frac{2}{n+1} \sum_{k=1}^{\infty} \frac{n}{(2n+k+1)(n+k+1)} + \\ & \frac{2}{n+2} \sum_{k=1}^{\infty} \frac{n}{(2n+k+2)(n+k+2)} + \\ & \frac{2}{n+3} \sum_{k=1}^{\infty} \frac{n}{(2n+k+3)(n+k+3)} + \cdots \\ &= \sum_{j=1}^{\infty} \frac{2}{n+j} \sum_{k=1}^{\infty} \frac{n}{(2n+k+j)(n+k+j)}. \end{aligned} \tag{40}$$

Owing to

$$\frac{1}{n+k+j} - \frac{1}{2n+k+j} = \frac{n}{(2n+k+j)(n+k+j)}$$

we can write

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{n}{(2n+k+j)(n+k+j)} &= \sum_{k=1}^{\infty} \frac{1}{n+k+j} - \sum_{k=1}^{\infty} \frac{1}{2n+k+j} \\ &= \sum_{k=1}^{n} \frac{1}{n+k+j} + \sum_{k=n+1}^{\infty} \frac{1}{n+k+j} - \sum_{k=1}^{\infty} \frac{1}{2n+k+j} \\ &= \sum_{k=1}^{n} \frac{1}{n+k+j} + \sum_{k=1}^{\infty} \frac{1}{2n+k+j} - \sum_{k=j+1}^{\infty} \frac{1}{2n+k+j} \\ &= \sum_{k=1}^{n} \frac{1}{n+k+j}. \end{aligned}$$

The series (40) gets now the form

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{2}{n+j}\sum_{k=1}^{\infty}\frac{n}{(2n+k+j)(n+k+j)} &= \sum_{j=1}^{\infty}\frac{2}{n+j}\sum_{k=1}^{n}\frac{1}{n+j+k}\\ &= 2\sum_{k=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(n+j)(n+j+k)}\\ &= 2\sum_{k=1}^{n}\left[\sum_{j=n+1}^{\infty}\frac{1}{j(j+k)}\right]\\ &= 2\sum_{k=1}^{n}\left[\frac{1}{k}\sum_{j=1}^{\infty}\frac{k}{j(j+k)}-\sum_{j=1}^{n}\frac{1}{j(j+k)}\right].\end{aligned} \tag{41}$$

Now the Psi function (Abramowitz and Stegun, 1972)

$$\psi(1+z) = -\gamma + \sum_{j=1}^{\infty}\frac{z}{j(j+z)}, \qquad z \neq -1,-2,-3,\ldots,$$
$$\psi(1) = -\gamma,\ \psi(n) = -\gamma + \sum_{j=1}^{n-1}\frac{1}{j}, \qquad n \geq 2,$$

where γ is the Euler-Mascheroni constant, gives

$$\sum_{j=1}^{\infty}\frac{k}{j(j+k)} = \gamma + \psi(1+k) = \sum_{j=1}^{k}\frac{1}{j}.$$

Applying this to (41) we get

$$\beta^2 Var\left(M_{(n)} \mid m_{\max} = \infty\right) = 2\sum_{k=1}^{n}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{j(k+j)}\right].$$

We will show now by induction that

$$2\sum_{k=1}^{n}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j} - \sum_{j=1}^{n}\frac{1}{j(k+j)}\right] = \sum_{k=1}^{n}\frac{1}{k^2}. \tag{42}$$

Let $n = 1$. Then we have

$$2\sum_{k=1}^{1}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j} - \sum_{j=1}^{1}\frac{1}{j(k+j)}\right] = 1 = \sum_{k=1}^{1}\frac{1}{k^2}.$$

Let assume now that (42) holds in the case n. Then at $n+1$ we have

$$2\sum_{k=1}^{n+1}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}-\sum_{j=1}^{n+1}\frac{1}{j(k+j)}\right]=2\sum_{k=1}^{n}\left[\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}-\sum_{j=1}^{n}\frac{1}{j(k+j)}\right]$$

$$-2\sum_{k=1}^{n}\frac{1}{(n+1)(k+n+1)}+2\left[\sum_{j=1}^{n+1}\frac{1}{j(n+1)}-\sum_{j=1}^{n+1}\frac{1}{j(n+1+j)}\right]$$

$$=\sum_{k=1}^{n}\frac{1}{k^2}+2\left\{\sum_{j=1}^{n+1}\frac{1}{(n+1)(n+1+j)}-\sum_{j=1}^{n}\frac{1}{(n+1)(n+1+j)}\right\}$$

$$=\sum_{k=1}^{n}\frac{1}{k^2}+\frac{1}{(n+1)^2}$$

$$=\sum_{k=1}^{n+1}\frac{1}{k^2}$$

This completes the demonstration that

$$Var\left(M_{(n)}\mid\infty\right)=\frac{1}{\beta^2}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{H_n^{(2)}}{\beta^2}$$

where $H_n^{(2)}$ is a harmonic number of order 2.

To the variance holds

$$\beta^2 Var\left(M_{(\infty)}\mid\infty\right)=\sum_{k=1}^{\infty}\frac{1}{k^2}=\zeta(2)=\frac{\pi^2}{6},$$

where $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ is a Riemann's zeta function (Abramowitz and Stegun, 1972). We see that for all $m_{\max}\in[m_{\min},\infty]$ and $n=1,2,3,\ldots$ we will have

$$Var\left(M_{(n)}\mid m_{\max}\right)\le\frac{1}{\beta^2}\sum_{k=1}^{n}\frac{1}{k^2}<\frac{\pi^2}{6\beta^2}\approx\frac{1.6449}{\beta^2}.$$

so the variance is a bounded function.

References:

Aki, K. (1965). Maximum likelihood estimate of b in the formula $\log N=a-bM$ and its confidence limits, *Bull. Earthquake Res. Inst., Tokyo Univ.* **43,** 237-239.

Cao, A. M. and Gao, S. S. (2002). Temporal variation of seismic *b*-values beneath northeastern Japan island arc, *Geophys. Res. Lett.* **29 (9)** 1–3.

Cosentino, P., and D. Luzio (1976). A generalization of the frequency-magnitude relation in the hypothesis of a maximum regional magnitude, *Ann. Geofis. (Rome)* **29**, 1-2, 3-8.

Cosentino, P., V. Ficara, and D. Luzio (1977). Truncated exponential frequency-magnitude relationship in the earthquake statistics, *Bull. Seism. Soc. Am.* **67,** 1615-1623.

Haarala, M. and L. Orosco (2016). Analysis of Gutenberg-Richter b-value and $m_{\max}$. Part I: Exact solution of Kijko-Sellevoll estimator $m_{\max}$. *Cuaderno de Ingeniería. Nueva Serie.* [Salta, Argentina], núm 9, 2016: 51-67.

Hamilton, R. M. (1967). Mean magnitude of an earthquake sequence*, Bull. Seism. Soc. Am*. **57,** 1115-1116.

Kijko, A., and G. Graham (1998). ''Parametric-historic'' procedure for probabilistic seismic hazard analysis. Part I: Assessment of maximum regional magnitude m_{max}, *Pure Appl. Geophys.* **152,** 413–442.

Kijko, A. (2004), Estimation of the maximum earthquake magnitude, $m_{\max}$, *Pure Appl. Geophys.* **161,** 1–27.

Page, R. (1968). Aftershocks and microaftershocks of the great Alaska earthquake of 1964, *Bull. Seism. Soc. Am.* **58,** 1131-1168.

Pisarenko, V. F., A. A. Lyubushin, V. B. Lysenko, and T. V. Golubieva (1996). Statistical estimation of seismic hazard parameters: Maximum possible magnitude and related parameters, *Bull. Seism. Soc. Am.* **86,** 691–700.

Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery (1992). *Numerical recipes in FORTRAN: The art of scientific computing*, 2$^{\text{nd}}$ ed., Cambridge University Press, Cambridge.

Rudin, W. (1987). *Principles of Mathematical Analysis,* 3$^{\text{rd}}$ ed., 14$^{\text{th}}$ printing, McGraw-Hill, Singapore.

Utsu, T. (1965). A method for determining the value of b in a formula $\log n = a - bM$ showing the magnitude-frequency relation for earthquakes, *Geophys. Bull. Hokkaido Univ.* **13,** 99-113.

Villarino, M. B. (2008). Ramanujan's Harmonic Number Expansion into Negative Powers, *J. Inequal. Pure and Appl. Math.,* **9 (3)**, Art. 89, 12pp.