Cuadernos de la Facultad de Ingeniería
Universidad Cátolica de Salta, vol. 8, 2014: 23-36

On the variational derivation of boundary value problems in the dynamics of structural elements

Ricardo Oscar Grossi

Summary

The calculus of variations is an old branch of mathematical analysis concerned with the problem of extremizing functionals, a generalization of the problem of finding extremes of functions of several variables. This discipline has a long history of interaction with other fields of mathematics and physics, particularly with mechanics. Engineers and applied mathematicians have increasingly used the techniques of calculus of variations to solve a large number of problems. Nevertheless, in this discipline the «operator» δ has been assigned special properties and handled using heuristic procedures. A mechanical «δ-method» has been developed and extensively used, as can be observed in the current engineering literature.
The objective of this paper is to present a rigorous formalism for the determination of boundary value problems which describe the static or dynamic behavior of structural elements. A discussion about the shortcomings of the use of the vague mechanical δ-method is presented.
Keywords: Variational calculus-rigorous formalism- functional-admissible directions

1. Introduction

The calculus of variations is a branch of mathematics concerned with extreme values in certain function spaces. It determines necessary conditions for a class of functions in order to extremize a given functional. These conditions are formulated in terms of ordinary differential equation or partial differential equations, boundary conditions and transition conditions. For centuries scientists have tried to formulate laws of natural sciences as extreme problems and called these laws variational principles. For this reason, in solid mechanics, the principle of virtual work and the Hamilton's principle provide straightforward methods for determining the differential equations of equilibrium and motion, boundary conditions and transition conditions. It is well known that there are two basic approaches to deriving the equations of motion

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of a mechanical system. One approach uses Newton's laws through an establishment of all the forces and moments in the system. The other is based on the application of Hamilton's principle. For complicated mechanical systems, the first procedure becomes intractable, and it is difficult to determine the type of boundary conditions and / or transitions conditions to be used in conjunction with the derived differential equations. On the other hand, the variational approach is very straightforward since variations of the kinetic and potential energies are utilized. This is one of the reasons why engineers, physicists and applied mathematicians are increasingly using techniques of calculus of variations to solve a large number of problems. The applications of this discipline now embrace a great variety of fields. The calculus of variations and the optimal control theory are widely used in biology, economics, astronautics, quantum mechanics, finance, etc. Nevertheless, calculus ofvariations is a discipline in which the «operator» δ has been assigned special properties not analyzed in the rigorous formalism of mathematics and a mechanical «δ -method» has been developed and extensively used.

Diverse opinions regarding the role of applied mathematics have been expressed and one approach is based on the use of pure mathematics with the field of application as an extension occupying a secondary role. Nevertheless, it is obvious that generally, the applied mathematician does not need to know concepts and theories as much as the pure mathematician does, but he should have good training in basic pure mathematics and should know the foundations of the relevant mathematical tools he is using in the solution of his problems, which have often emerged from real-world situations. It is not true that the mathematical theory needed by applied mathematicians is remote from the urgent problems that arise in various fields of engineering and applied science. Professor Richard Courant [1] remarked: «Pure mathematicians sometimes are satisfied with showing that the non-existence of a solution implies a logical contradiction, while engineers might consider a numerical result as the only reasonable goal. Such one sided views seem to reflect human limitation rather than objective values. In itself mathematics is an indivisible organism uniting theoretical contemplation and active application».

In calculus, real valued functions defined on certain subsets of the n -dimensional Euclidean space $\mathbb{R}^n$, are used. The determination of extreme values of a function $f : D \to \mathbb{R},\ D \subseteq \mathbb{R}^n$, is concerned with finding elements of D with which the smallest (largest) value of f is associated. A decisive role in the optimization of this type of functions is played by its partial derivatives or more generally by its directional derivatives. It is commonly accepted that the concept of *functional* is a natural generalization of the concept of function given in elementary calculus. Since the calculus of variations is concerned with the problem of extremizing functionals, it is natural to consider this problem as a generalization of the problem of finding extremes of real valued functions of several variables. While it might seem that the introduction of the concept of *variation* of a functional should be subsumed into the mentioned rigorous procedure, this is not the case. Thus, a number of books and papers have been published dealing with the calculus of variations and particularly with the definition of variation of a functional, from a heuristic point of view. For this purpose, a vague and obscure procedure based on an analogy between the variational operator δ and the differential operator d of functions is adopted.

It is true that since the calculus of variations has called the attention of several mathematicians, who made important contributions to its development, there are many technical details which are hardly available to a non-mathematician. But fortunately, it is possible to present a minimal set of basic concepts of this discipline, using only certain abstractions of what are considered to be simple ideas from elementary calculus. In this aspect, the elementary functional analysis provides a much better and deeper understanding of the fundamental concepts of: *variation of a functional*, *space of admissible functions, space of admissible directions*, and *weak* and *strong local extremes*.

Professor Magnus Hestenes claimed that «there is no discipline in which more correct results can be obtained by incorrect means than in the calculus of variations», [2]. This dictum of a prestigious specialist emphasizes the importance of the use of rigorous formalisms, rather than obscure heuristic definitions.

The primary purpose of this paper is to make a small contribution toward reducing the gap between the abundance of concepts and methods available in abstract calculus of variations and their limited use in various areas of vibrations of structures. For this purpose, a rigorous procedure for the determination of boundary value problems, which describe the statical or dynamical behavior of a common structural element, is discussed.

Substantial literature has been devoted to the formulation - by means of the calculus of variations - of boundary value problems in the statics and dynamics of mechanical systems. It is not the intention to review the literature; consequently, only some of the relevant works will be cited. A number of textbooks, [3-13] deal with the classical variational calculus and others, [14-24] include rigorous treatments of the theoretical aspects of this discipline. Several textbooks, [25-30] also present formulations, by means of variational techniques, of boundary value problems in the statics and dynamics of beams, frames and plates.

A secondary purpose of this paper is to present a rigorous variational formulation to determine the boundary value problems which describe the dynamical behavior of a freely vibrating beam. For this purpose, the construction of the domain and space of admissible directions, which corresponds to the variation of the functional which in mechanics is called *action integral* is included. In addition, the presence of some errors in the literature, and particularly in the formulation of fundamental lemma of the calculus of variations is also demonstrated.

This paper is organized in the following way. In Section 2 some basic concepts are treated. In Section 3 a discussion about the concept of variation of a functional, which covers both the heuristic and the rigorous form, is included. In Section 4 the Hamilton's principle is rigorously stated in the case of a freely vibrating beam. Finally, Section 5 contains the conclusions of this paper.

2. Some basic topics

It is commonly accepted that the concept of functional is a generalization of that of a real function of real variable and the following rigorous definition can be found even in engineering textbooks.

Definition 1. Let D be a subset of a linear space V. A mapping which assigns to each element of D exactly one real number is called a *functional* defined in D, and it is denoted by

$I: D \to \mathbb{R}$. A typical example is

$$I(u) = \int_a^b F\left(x, u(x), u'(x)\right) dx, \tag{1}$$

defined in $C^1[a,b]$, the space of all real valued functions with a continuous derivative on the interval $[a,b]$.

Remark 1. Definition 1 illustrates the point of view of the functional analysis. In calculus, the notion of a real valued function of a real variable is associated with the real numbers which constitute its values, but the functional analysis view is that it defines a correspondence between pairs of elements of prescribed sets. The concepts of *linear* or *vectorial space* and *normed space* are rather intuitive and can be presented as natural generalizations of the corresponding definitions in the Euclidean space $\mathbb{R}^n$. These generalized notions are applied throughout mathematics, science and engineering [23], [31], [32].

3. The first variation of a functional

3.1 Heuristic development

As stated above, a number of books and papers have appeared which treat the calculus of variations from a heuristic point of view using a vague and obscure procedure based on an analogy between the variational operator δ and the differential operator d of functions. The following statements have been compiled from some textbooks included in the reference list:

In the calculus of variations it is a common practice to use δu to denote what is defined as the *first variation* of the function u, which is given by

$$\delta u = \varepsilon v, \tag{2}$$

where ε is a *small* arbitrary real number and v an arbitrary function. Thus δu is considered as an operator that changes from the function u into δu. The derivatives are changed in the same form. For instance, du / dx is changed into

$$\delta\left(\frac{du}{dx}\right) = \varepsilon \frac{dv}{dx}. \tag{3}$$

The variational operator can be interchanged with derivatives and integrals. For instance,

$$\delta \int_\Omega F dx = \int_\Omega \delta F dx. \tag{4}$$

In analogy with the concept of total differential dF of a real function of several variables $F = F(x,y,z)$ given by

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz,$$

the variational operator δ acts like the total differential defined above. In consequence, the first variation of $F = F(x,u,u')$ is defined by

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'. \tag{5}$$

Finally, in the case of the functional given by (1), the use of property (4) leads to

$$\delta I(u) = \int_a^b \delta F(x,u,u') dx. \tag{6}$$

3.2. Rigorous definitions

Within reasonable limits, the arguments from the extreme values theory of real valued functions of several variables, find their counterpart in the theory of extremes values of functionals. Thus, the concept of the variation of a functional can be easily stated as a generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$. This procedure should be the key to eliminate the lengthy and obscure definition of the variation of a functional using the Eqs. (2) to (6). Let us recall the definition of directional derivative:

Suppose we are given a real valued function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}^n$. If $\mathbf{x}$ is an interior point of S and $\mathbf{v} \in \mathbb{R}^n$ an arbitrary vector of unit length $\left(\|\mathbf{v}\| = 1\right)$, then the directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{v}$ is given by

$$f'\left(\mathbf{x}, \mathbf{v}\right) = \lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{v}) - f(\mathbf{x})}{\varepsilon},$$

if this limit exists.

If I is a functional defined in a subset D of a vectorial space V, its directional derivative (called variation) is easily furnished by a straightforward generalization of the above definition of directional derivative of a function.

Definition 2. Let I be a functional defined in a subset D of a vectorial space V. If $u \in D$ and $v \in V$, the variation of I in the point u and in the direction v, is given by

$$\delta I\left(u; v\right) = \lim_{\varepsilon \to 0} \frac{I\left(u + \varepsilon v\right) - I\left(u\right)}{\varepsilon} = \frac{dI}{d\varepsilon}\left(u + \varepsilon v\right)\Big|_{\varepsilon = 0}, \tag{7}$$

when the ordinary derivative with respect to the real variable ε exists at $\varepsilon = 0$.

Since the application of (7) requires deriving with respect to ε under the integral sign, in the case of the functional defined by (1) we should require that the function $F = F\left(x, u, w\right)$ has continuous partial derivatives and $u \in C^1\left[a, b\right]$,; then, we have

$$\begin{aligned}
&\delta I\left(u; v\right) = \int_a^b \frac{\partial}{\partial \varepsilon} F\left(x, u\left(x\right) + \varepsilon v\left(x\right), u'\left(x\right) + \varepsilon v'\left(x\right)\right) dx\Big|_{\varepsilon = 0} = \\
&= \int_a^b \left(\frac{\partial F}{\partial u}\left(x, u\left(x\right) + \varepsilon v\left(x\right), u'\left(x\right) + \varepsilon v'\left(x\right)\right) v\left(x\right) + \right. \\
&\left. + \frac{\partial F}{\partial w}\left(x, u\left(x\right) + \varepsilon v\left(x\right), u'\left(x\right) + \varepsilon v'\left(x\right)\right) v'\left(x\right)\right) dx\Big|_{\varepsilon = 0} = \\
&= \int_a^b \left(\frac{\partial F}{\partial u}\left(x, u\left(x\right), u'\left(x\right)\right) v\left(x\right) + \frac{\partial F}{\partial w}\left(x, u\left(x\right), u'\left(x\right)\right) v'\left(x\right)\right) dx.
\end{aligned} \tag{8}$$

The above is well known, at least heuristically, to anyone who works in the field of calculus of variations.

3.3 Admissible directions

In definition 2, it can be noted that element v, which generalizes the concept of direction, is simply an element of the vector space V. It plays an essential role in the minimization of a functional. In this process we are interested in those functions u and directions v, in which the variation of I exists. For instance, if we want to find a function $u \in C^1\left[a,b\right]$ so that the functional (1) assumes a minimum where by

$$u\left(a\right) = A, \qquad u\left(b\right) = B, \tag{9}$$

are given, we are not interested in all functions $u \in C^1\left[a,b\right]$ but only in those which satisfy the conditions (9). On the other hand, we are interested in considering for each $u \in C^1\left[a,b\right]$ those directions $v \in V$ in which the functional I admits the variation $\delta I\left(u;v\right)$ as is stated in the following definition.

Definition 3. A direction $v \in V$ is admissible if:

$\left(i\right)$ $u + \varepsilon v \in D, \forall \varepsilon$ sufficiently small,
$\left(ii\right)$ $\delta I\left(u;v\right)$ exists.

The space of admissible directions is commonly denoted by D_a.

Remark 2. It must be noted that there is no need to introduce the concept of *variation* δu of the actual configuration u which usually is presented in the following form:

«Suppose $u\left(x\right)$ is indeed the function of x which gives (1) a minimum value, and $u^*\left(x\right)$ is a second function of x which is *at most infinitesimally different from* $u\left(x\right)$ at every point x within the interval $\left[a,b\right]$. Define

$$\delta u\left(x\right) = u^*\left(x\right) - u\left(x\right).$$

The variation of a function should be understood to represent an infinitesimal change in the function u at a given value of x. The change is arbitrary; that is, it is a virtual change.»

This lengthy and obscure definition should be avoided because while no advantage is taken of its use, a source of confusion is eliminated. Although it is an ordinary function, in mechanics, it is traditional to denote by δu an admissible virtual displacement of u, . It is particularly used in the powerful virtual work principle [24].

3.4 Necessary condition for an extreme

When a real valued function $f : S \to \mathbb{R}$ defined on a set $S \subset \mathbb{R}^n$, has a local extremal point $\mathbf{x_0} \in S$ in which f has continuous partial derivatives, then

$$f'\left(\mathbf{x}_0, \mathbf{v}\right) = 0,$$

for each vector $\mathbf{v} \in \mathbb{R}^n$ of unit length. In the context of functionals, the following theorem can be demonstrated. See for instance, references [18], [19].

Theorem 1. Let $\left(V, \left\|\bullet\right\|\right)$ be a normed space and $I : D \to \mathbb{R}$, where $D \subset V$. If the functional I assumes a local extremum at $u_0 \in D$, then

$$\delta I\left(u_0, v\right) = 0, \forall v \in D_a. \tag{10}$$

Remark 3. It must be noted that the condition (10) requires the use of all admissible directions and generally there may be enough directions to permit this condition to determine the function u_0. This is consistent with the fundamental lemma which must be applied to obtain a more useful condition than (10).

The fundamental lemma

If $F \in C[a,b]$ and $\int_a^b F(x)v(x)dx = 0$, for any **arbitrary continuous** function v, which verifies $v(a) = v(b) = 0$ for all $x \in (a,b)$, then $F \equiv 0$ on (a,b).

4. The Euler-Lagrange equation

4.1 Heuristic development

The following statements have been compiled from some textbooks of the reference list:

«The necessary condition for the functional $I(u) = \int_a^b F(x,u,u')\,dx$, to have a minimum is $\delta I = 0$, so we have

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dx = 0. \tag{11}$$

Since we cannot use the fundamental lemma because (11) is not in the adequate form, we integrate the second term by parts and obtain

$$\int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' \right) dx = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right) \delta u dx + \left. \frac{\partial F}{\partial u'}\delta u \right|_a^b = 0. \tag{12}$$

In the case of fixed ends all admissible variations must satisfy the conditions: $\delta u(a) = \delta u(b) = 0$, then (12) reduces to

$$\int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) \right) \delta u dx = 0, \forall \delta u \quad \text{in} \quad (a,b). \tag{13}$$

In consequence, if the fundamental lemma is applied to (13) with $v = \delta u$, we obtain

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'} \right) = 0, \forall x \in (a,b). \tag{14}$$

4.2 Rigorous formalism

Instead of the functional of the preceding discussions, let us consider the more interesting functional which corresponds to a freely vibrating beam.

Let us consider a uniform beam of length l, rigidly clamped at both ends and which executes transverse vibrations when subjected to an external load of density $q = q(x,t)$. We suppose that the vertical position of the beam at any time t is given by the function

$$w = w\left(x,t\right), \forall x \in [0,l].$$

It is well known that at time t the kinetic energy and the total potential energy due to the elastic deformation of the beam and the potential energy of the external load are respectively given by

$$E_c = \frac{1}{2}\int_0^l \rho A\left(\frac{\partial w}{\partial t}\left(x,t\right)\right)^2 dx, \tag{15}$$

and

$$E_p = \frac{1}{2}\int_0^l \left(EI\left(\frac{\partial^2 w}{\partial x^2}\left(x,t\right)\right)^2 - 2q\left(x,t\right)w\left(x,t\right)\right)dx, \tag{16}$$

where ρ is the mass per unit length, A the cross-sectional area, and EI the flexural rigidity of the beam.

Hamilton's principle requires that between times t_0 and t_1, at which the positions are known, the motion will make stationary the action integral

$$I\left(u\right) = \int_{t_0}^{t_1} \left(E_c - E_p\right)dt,$$

on the space of admissible functions. Hence, from (15) and (16) we have

$$I\left(u\right) = \frac{1}{2}\int_{t_0}^{t_1}\int_0^l \left[\rho A\left(\frac{\partial w}{\partial t}\right)^2 - EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2qw\right]dxdt. \tag{17}$$

In order to make the mathematical developments required by the use of the applications of the techniques of the calculus of variations, we assume that $w \in C^4\left(\overline{G}\right)$, where $\overline{G} = \left[0,l\right]\times\left[t_0,t_1\right]$. Since the beam is rigidly clamped, the boundary conditions are given by

$$w\left(0,t\right) = 0,\ w\left(l,t\right) = 0,\ \forall t \geq 0, \tag{18a,b}$$

$$\frac{\partial w}{\partial x}\left(0,t\right)=0,\ \frac{\partial w}{\partial x}\left(l,t\right)=0,\ \forall t\geq 0. \tag{19a,b}$$

In view of these observations and since Hamilton's principle requires that at times t_0 and t_1 the positions are known, the domain of the functional (17) is given by

$$\begin{aligned} D^{C,C} &= \left\{w; w\in C^4\left(\bar{G}\right), w\left(0,t\right)=w\left(l,t\right)=w_x\left(0,t\right)=w_x\left(l,t\right)=\right.\\ &\left.=0,\forall t\in\left[t_0,t_1\right], w\left(x,t_0\right)=h_0\left(x\right), w\left(x,t_1\right)=h_1\left(x\right),\forall x\in\left[0,l\right]\right\}, \end{aligned} \tag{20}$$

where h_0 and h_1 denote the functions which give the positions of the beam at t_0 and t_1 and a non-standard notation has been implemented in order to handle the spaces of admissible functions and directions effectively. Thus, the superscripts in (20) are consistent with the ends conditions. From definition 3, it follows that the corresponding space of admissible directions is given by

$$\begin{aligned} D_a^{C,C} &= \left\{v; v\in C^4\left(\bar{G}\right), v\left(0,t\right)=v\left(l,t\right)=v_x\left(0,t\right)=\right.\\ &\left.=v_x\left(l,t\right)=0,\forall t\in\left[t_0,t_1\right], v\left(x,t_0\right)=v\left(x,t_1\right)=0,\forall x\in\left[0,l\right]\right\}. \end{aligned} \tag{21}$$

To see this, we only have to note that for arbitrary $w\in D^{C,C}$ and arbitrary direction $v\in D_a^{C,C}$ it is true that $w+\varepsilon v\in D^{C,C}$, too. The condition $\left(ii\right)$ of definition 3 is satisfied if $w,v\in C^4\left(\bar{G}\right)$ and $q\in C\left(\bar{G}\right)$. Now, in the case of the functional given by (17), the condition of stationary functional is given by

$$\delta I\left(w^{C,C};v\right)=0,\forall v\in D_a^{C,C}. \tag{22}$$

If $w,v\in C^2(\bar{G})$ the application of definition 2 leads to

$$\begin{aligned} \delta I\left(w;v\right) &= \frac{d}{d\varepsilon}I\left(w+\varepsilon v\right)\Big|_{\varepsilon=0} =\\ &= \int_{t_0}^{t_1}\int_0^l\left(\rho A\frac{\partial w}{\partial t}\frac{\partial v}{\partial t}-EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 v}{\partial x^2}+qv\right)dxdt, \end{aligned} \tag{23}$$

where $w=w^{C,C}$.

Let us consider the first term in (23). Since $w,v\in C^2(\bar{G})$ we can integrate by parts with respect to t and if we apply the conditions $v\left(x,t_0\right)=v\left(x,t_1\right)=0,\forall x\in\left[0,l\right]$, imposed in (21) we obtain

$$\int_{t_0}^{t_1}\int_0^l \rho A \frac{\partial w}{\partial t}\frac{\partial v}{\partial t}dxdt = -\int_{t_0}^{t_1}\int_0^l \rho A \frac{\partial^2 w}{\partial t^2} vdxdt.$$

In an analog situation since $w, v \in C^4\left(\overline{G}\right)$ we can integrate by parts twice with respect $x,$ to thus obtaining

$$\begin{aligned}\int_{t_0}^{t_1}\int_0^l EI\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 v}{\partial x^2}dxdt =& \int_{t_0}^{t_1}\int_0^l \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right)vdxdt + \\ +\int_{t_0}^{t_1}&\left(-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)v\Big|_0^l + EI\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial x}\Big|_0^l\right)dt.\end{aligned} \tag{25}$$

By replacing (24) and (25) into (23), we have

$$\begin{aligned}\delta I\left(w;v\right) =& \int_{t_0}^{t_1}\int_0^l\left(-\rho A\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + q\right)vdxdt + \\ +\int_{t_0}^{t_1}&\left(\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)v\Big|_0^l - EI\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial x}\Big|_0^l\right)dt.\end{aligned} \tag{26}$$

According to (21) and (26), the condition (22) reduces to

$$\begin{aligned}&\delta I\left(w;v\right) = \int_{t_0}^{t_1}\int_0^l\left(-\rho A\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) + q\right)vdxdt = 0, \\ &, \forall v \in D_a^{E,E},\end{aligned} \tag{27}$$

where $w = w^{C,C}$.

Now the application the fundamental lemma of calculus of variations in $\mathbb{R}^n$, it follows that the function $w^{C,C}$ must satisfy the differential equation

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\left(x,t\right)\right) + \rho A\frac{\partial^2 w}{\partial t^2}\left(x,t\right) = q\left(x,t\right),\ \forall x \in \left(0,l\right), \forall t > 0. \tag{28}$$

It has been demonstrated that the boundary value problem which corresponds to a vibrating beam rigidly clamped is given by the differential equation (28) and the boundary conditions (18)-(19).

Now, let us assume that the beam is simply supported at both ends. In the manner of achieving the spaces (20) and (21) we have that, in this case, the spaces of admissible functions and directions are respectively given by

$$\begin{aligned} D^{S,S} &= \left\{w; w \in C^4\left(\bar{G}\right), w\left(0,t\right) = w\left(l,t\right) = 0, \forall t \in \left[t_0, t_1\right],\right. \\ & \left. w\left(x,t_0\right) = h_0\left(x\right), w\left(x,t_1\right) = h_1\left(x\right),\ \forall x \in \left[0,l\right]\right\}, \end{aligned} \tag{29}$$

$$\begin{aligned} D_a^{S,S} &= \left\{v; v \in C^4\left(\bar{G}\right), v\left(0,t\right) = v\left(l,t\right) = 0, \forall t \in \left[t_0, t_1\right],\right. \\ & \left. v\left(x,t_0\right) = v\left(x,t_1\right) = 0, \forall x \in \left[0,l\right]\right\}. \end{aligned} \tag{30}$$

Now, the condition of stationary functional is given by

$$\delta I\left(w^{S,S}; v\right) = 0, \forall v \in D_a^{S,S},$$

and by virtue of the inclusion $D_a^{E,E} \subset D_a^{S,S}$, we have

$$\delta I\left(w^{S,S}; v\right) = 0, \forall v \in D_a^{C,C},$$

from which it follows that the function $w^{S,S}$ must satisfy the differential equation (28).

By replacing w by $w^{S,S}$ in Eq. (26) and using directions from the space (30), the condition (22) reduces to

$$\delta I\left(w; v\right) = \int_{t_0}^{t_1}\left(EI\frac{\partial^2 w}{\partial x^2}\frac{\partial v}{\partial x}\bigg|_0^l\right)dt = 0, \forall v \in D_a^{S,S} \text{con} \quad w = w^{S,S}$$

In the manner of achieving (28) we have that the function $w^{S,S}$ must satisfy the differential equation (28), the geometric boundary conditions (18 a, b) and the natural boundary conditions

$$EI\left(0\right)\frac{\partial^2 w}{\partial x^2}\left(0,t\right) = 0, \quad EI\left(l\right)\frac{\partial^2 w}{\partial x^2}\left(l,t\right) = 0, \forall t \geq 0. \tag{32 a, b}$$

If the beam is free at both ends, we must consider the condition

$$\delta I\left(w^{F,F}; v\right) = 0, \forall v \in D_a^{F,F},$$

where

$$D_a^{F,F} = \left\{ v; v \in C^4\left(\bar{G}\right), v\left(x, t_0\right) = v\left(x, t_1\right) = 0, \forall x \in \left[0, l\right] \right\}. \tag{33}$$

In the manner of achieving the previous boundary value problems, we have that the function $w^{F,F}$ must satisfy the differential equation (28) and the natural boundary conditions

$$EI\left(0\right)\frac{\partial^2 w}{\partial x^2}\left(0,t\right) = 0,\ EI\left(l\right)\frac{\partial^2 w}{\partial x^2}\left(l,t\right) = 0, \forall t \geq 0, \tag{34}$$

$$\frac{\partial}{\partial x}\left(EI\left(0\right)\frac{\partial^2 w}{\partial x^2}\left(0,t\right)\right) = 0, \frac{\partial}{\partial x}\left(EI\left(l\right)\frac{\partial^2 w}{\partial x^2}\left(l,t\right)\right) = 0, \forall t \geq 0. \tag{35}$$

The remaining boundary conditions are obtained as a combination of the analyzed cases.

5. Concluding Remarks

It has been demonstrated that the use of the mechanical «δ -method» is not necessary since it is a source of confusion and its lack of rigour leads to obscure definitions. Moreover, it is more natural and clearer to present the variation of a functional as a straightforward generalization of the definition of the directional derivative of a real valued function defined on a subset of $\mathbb{R}^n$. The determination of the space of admissible functions and the space of admissible directions generates a clear statement of the problem. This is particularly true in the study of the dynamical behaviour of structural systems. This has been shown in Section 4.

Surely, opinions will express that the heuristic procedure described in Section 4.1 finally leads to the same correct results of Section 4.2. However, the use of functional analysis leads to a deeper and clearer understanding of the problem. Today, solving practical problems necessitates the introduction of sophisticated mathematical tools such as the concept of weak solution and Sobolev Spaces. Emphasis should be placed on the use of abstract results because despite of the abstractness of these topics, they lead to very practical outcomes. For instance, the finite element method is a powerful computational technique for the solution of boundary value problems that arise in various fields of engineering and applied science. It is necessary to use the Sobolev spaces to know the qualities of the numerical approximation of the mentioned method, [24], [31]-[33].

There exists a growing gap between pure mathematicians and applied scientists to the point that experts in the two mentioned areas are unable to understand and to communicate. It is impossible to reduce, or at least to stop this gap, if heuristics and obscure mathematical procedures are used. For instance, from some textbooks the following statements have been compiled:

Definition

«Mathematically, a functional is *a real number* obtained by operating on functions from a given set».

Lemma.

$$\textit{If } \ F \in C\left[a,b\right] \ \textit{ and } \int_a^b F\left(x\right) v\left(x\right) dx = 0,$$

for any *arbitrary continuous* function v, for all $x \in \left(a,b\right)$, then $F \equiv 0$ on $\left(a,b\right)$.»

It is true that this lemma can be proved without the usual restrictions $v \in C\left[a,b\right]$, $v\left(a\right) = v\left(b\right) = 0$, but then it cannot be used in a problem which involves fixed end points because in this case the admissible directions v are functions which vanish at the endpoints a and b as in the case treated in Section 4.1. It must be noted that in the definition of variation (3.1) ε is a *small* arbitrary real number. Moreover, δu satisfies the conditions: $\delta u\left(a\right) = \delta u\left(b\right) = 0$, so it is not an *arbitrary continuous* function v for all $x \in \left(a,b\right)$, as is required in the above lemma .

This type of imprecision could be originated in the use of obscure and vague concepts which can be avoided using only certain abstractions of what are considered to be simple ideas from elementary calculus.

Finally, it is emphasized that the rigorous procedure described is particularly adequate to derive the boundary value problems of beams with internal hinges and plates with a line hinge. In these cases, the first derivatives of the deflection functions are not continuous (in the points where the hinges are located) and the analytical developments require a careful analysis of the regularity properties of the admissible functions.

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Acknowledgments

The present investigation was sponsored by Consejo de Investigaciones de la de la Universidad Católica de Salta. Project Nº 115-13.