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Ramanujan's Expansion for the real valued Harmonic Numbers

Expansión de Ramanujan para Números Armónicos reales

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Abstract

The Ramanujan's Harmonic Number Expansion plays an important role to accelerate the calculus of the Kijko-Sellevoll function 3 (KS-3) (Haarala and Orosco, 2016). It gives much better performance against the ready-made algorithms. In this paper, we generalized the solution so that it holds also for real valued Harmonic Numbers, knowing that a solution for integer-valued numbers was given. It means that we can use Ramanujan's Harmonic Number. Expansion for $\sum_{j=1}^{k-1}(\eta+j)^{-1}$ when $\eta \in \mathbb{R}_+$.

Keywords: Kijko-Sellevoll function, Ramanujan's Harmonic Number Expansion.

Resumen

La expansión de números armónicos de Ramanujan, juega un importante papel para acelerar el cálculo de la Función 3 de Kijko-Sellevoll (Haarala y Orosco, 2016). Tiene un mejor comportamiento que los algoritmos propuestos. En este informe, generalizamos la solución para extender el método a números reales, en el conocimiento que fue propuesta una solución para números enteros. Esto implica que podemos usar la expansión por números armónicos de Ramanujan, para $\sum_{j=1}^{k-1}(\eta+j)^{-1}$, cuando $\eta \in \mathbb{R}_+$.

Palabras clave: Función de Kijko-Sellevoll, Expansión por números armónicos de Ramanujan.

1. Introduction

In a Kijko-Sellevoll function 3 (KS-3)

$$f_\eta^{KS-3}(x) = \sum_{k=2}^{\infty} \frac{2\eta}{2\eta+k} \left\{ \sum_{j=1}^{k-1} \frac{1}{\eta+j} \right\} \frac{\left(1-\exp[-x]\right)^k}{\eta+k},$$

we have a sum

$$\sum_{j=1}^{k-1} \frac{1}{\eta+j}, \tag{1}$$

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which can take a lot of time of calculation in common PC's, when k is big. (For example, if the van Wijngaarden transformation is used, it can reach up to $k = 2^{57} - 1$ terms (Haarala and Orosco, 2016).

We can see expression (1) as (Abramowitz and Stegun, 1972)

$$\sum_{j=1}^{k-1} \frac{1}{\eta + j} = \psi(\eta + k) - \psi(\eta + 1) = H_{\eta+k-1} - H_{\eta} . \tag{2}$$

Here real valued Harmonic Number H_η is defined as

$$H_\eta = \psi(\eta + 1) + \gamma = \sum_{k=1}^{\infty} \frac{\eta}{k(k+\eta)} = \int_0^1 \frac{1-t^\eta}{1-t} dt .$$

We found empirically that the Ramanujan's Harmonic Number Expansion (Haarala and Orosco, 2016)

$$\sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2}\log(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5} - \frac{191}{360360m^6} + \cdots , \tag{3}$$

where $m = n(n+1)/2$ is a triangular number and $\gamma = 0.57721566490153286$ is the Euler constant: (3) can be used instead of the MATLAB function named Harmonic also when n is real and it works at least 100 times quicker. In this work, we will show that (let $\mathfrak{M} = \eta(\eta+1)/2,\ \eta \in \mathbb{R}_+$)

$$H_\eta \sim \frac{1}{2}\log(2\mathfrak{m}) + \gamma + \frac{1}{12\mathfrak{m}} - \frac{1}{120\mathfrak{m}^2} + \frac{1}{630\mathfrak{m}^3} - \frac{1}{1680\mathfrak{m}^4} + \frac{1}{2310\mathfrak{m}^5} - \frac{191}{360360\mathfrak{m}^6} + \cdots .$$

This expansion has been a mystery and it seems that the Villarino's paper (2004) was the first published proof for the Expansion of (3). As Villarino (2008) wrote: "The origin of Ramanujan's formula is mysterious. Berndt (1998) notes that in his remarks. Our analysis of it is *a posteriori* and, although it is full and complete, it does not shed light on how Ramanujan came to think of his expansion". Ramanujan died on 1920.

We shall give short introduction to the Bernoulli's numbers and polynomials since they play a quite important role in asymptotic series, especially in the Euler-Maclaurin series. This introduction and more information can be found in Sabah and Gourdon (2002).

Jacob Bernoulli (1654-1705) was the first to give the formula for the Bernoulli's numbers. He found it by studying the sums of powers

$$\begin{aligned}\sum_{k=1}^{n-1} k^p &= \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k} \\ &= \frac{B_0}{0!} \frac{n^{p+1}}{p+1} + \frac{B_1}{1!} n^p + \frac{B_2}{2!} p n^{p-1} + \frac{B_3}{3!} p(p-1) n^{p-2} + \cdots + \frac{B_p}{1!} n .\end{aligned}$$

For example, we have ($p=1$)

$$1+2+3+\cdots+n-1=\frac{1}{2}n^2-\frac{1}{2}n=\frac{n(n-1)}{2}.$$

It is interesting to remark that the degree of these polynomials depends only on the power p, not on the number n of terms in the sum. The first Bernoulli's numbers are $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42$. Actually, it generally holds $B_{2k+1}=0$ for $k=1,2,3,\ldots$

Bernoulli polynomials $B_k(x)$ are defined by

$$\frac{z\,e^{zx}}{e^z-1}=\sum_{k=0}^{\infty}B_k(x)\frac{z^k}{k!}.$$

The first polynomials look like

$$\begin{aligned}
&B_0(x)=1\\
&B_1(x)=x-\frac{1}{2}\\
&B_2(x)=x^2-x+\frac{1}{6}\\
&B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x\\
&B_4(x)=x^4-2x^3+x^2-\frac{1}{30}\\
&\qquad\vdots
\end{aligned}$$

In the Bernoulli polynomials, we have $B_k=B_k(0)=B_k(1)$ for $k=0,2,3,4,\ldots$ and in the case of $k=1$ it is $B_1=B_1(0)=-B_1(1)$. That is to say, the Bernoulli numbers at 0 and 1 are the same except when $k=1$.

Some relations of those polynomials (Abramowitz and Stegun, 1972) are presented here:

$$\begin{aligned}
&(a)\quad |B_{2k}(x)|<|B_{2k}|, && k=1,2,3,\ldots,\quad 0<x<1,\\
&(b)\quad B_k\left(\frac{1}{2}\right)=-\left(1-2^{1-k}\right)B_k, && k=0,1,2,\ldots,\\
&(c)\quad \frac{2(2k)!}{(2\pi)^{2k}}<(-1)^{k+1}B_{2k}<\frac{2(2k)!}{(2\pi)^{2k}}\left(\frac{1}{1-2^{1-2k}}\right), && k=1,2,3,\ldots.
\end{aligned}\tag{4}$$

Considering the second equation (4)(b), we see that $B_{2k+1}(1/2)=0$ for all $k=0,1,2,\ldots$ $B_1(x)$ vanishes if and only if $x=1/2$ and the Euler-Maclaurin series has one term less (as we will see below).

The third equation (4)(c) shows that

$$0<\left(1-2^{1-2k}\right)\frac{2(2k)!}{(2\pi)^{2k}}<(-1)^k B_{2k}\left(\frac{1}{2}\right)<\frac{2(2k)!}{(2\pi)^{2k}},\qquad k=1,2,3,\ldots.$$

We can see also (from (4)(a-c)) that

$$0 \le \left(1-2^{1-2k}\right)\left|B_{2k}(x)\right| \underset{(*)}{\le} \left(1-2^{1-2k}\right)(-1)^{k+1} B_{2k} = (-1)^k B_{2k}\left(\frac{1}{2}\right), \quad k=1,2,3,\ldots, \ 0 \le x < 1,$$

where equality *(*)* holds at $x = 0$. Hence,

$$-(-1)^k B_{2k}\left(\frac{1}{2}\right) \le \left(1-2^{1-2k}\right) B_{2k}(x) \le (-1)^k B_{2k}\left(\frac{1}{2}\right), \quad k=1,2,3,\ldots, \ 0 \le x < 1. \tag{5}$$

One important application of the Bernoulli's numbers and polynomials is the Euler-Maclaurin formula (Abramowitz and Stegun, 1972)

$$\begin{aligned}\sum_{k=0}^{m} F(a+kh) = \frac{1}{h}\int_a^b F(t)\,dt + \frac{1}{2}\{F(b)+F(a)\} \\ + \sum_{k=1}^{n-1} \frac{h^{2k-1}}{(2k)!} B_{2k}\left\{F^{(2k-1)}(b) - F^{(2k-1)}(a)\right\} \\ + \frac{h^{2n}}{(2n)!} B_{2n} \sum_{k=0}^{m-1} F^{(2n)}(a+kh+\theta h),\end{aligned}$$

where $h=(b-a)/m$, $0<\theta<1$ and $F(x)$ has $2n$ continuous derivatives. Here the Bernoulli's numbers are adopted at $x=0$. Using this formula, it can be shown that

$$\sum_{k=1}^{n} \frac{1}{k} \sim \log(n) + \gamma - \sum_{k=1}^{p} \frac{B_k}{k\,n^k}.$$

A more general version of Euler-Maclaurin formula is (Abramowitz and Stegun, 1972)

$$\begin{aligned}\sum_{k=0}^{m-1} F(a+kh+\omega h) = \frac{1}{h}\int_a^b F(t)\,dt + \sum_{k=1}^{p} \frac{h^{k-1}}{k!} B_k(\omega)\left\{F^{(k-1)}(b) - F^{(k-1)}(a)\right\} \\ - \frac{h^p}{p!}\int_0^1 \hat{B}_p(\omega - t)\left\{\sum_{k=0}^{m-1} F^{(p)}(a+kh+th)\right\} dt\end{aligned}$$

where $h=(b-a)/m$, $0 \le \omega \le 1$, $p \le 2n$, $\hat{B}_k(x) = B_k(x-\lfloor x \rfloor)$ ($\lfloor x \rfloor$ means the largest integer $\le x$) and $F(x)$ has $2n$ continuous derivatives. If we set now $\omega = 1/2$, $p = 2r+2$, then we have

$$\begin{aligned}\sum_{k=0}^{m-1} F\left(a+kh+\frac{1}{2}h\right) = \frac{1}{h}\int_a^b F(t)\,dt + \sum_{k=1}^{r+1} \frac{h^{2k-1}}{(2k)!} B_{2k}\left(\frac{1}{2}\right)\left\{F^{(2k-1)}(b) - F^{(2k-1)}(a)\right\} \\ - \frac{h^{2r+2}}{(2r+2)!}\int_0^1 \hat{B}_{2r+2}\left(\frac{1}{2}-t\right)\left\{\sum_{k=0}^{m-1} F^{(2r+2)}(a+kh+th)\right\} dt\end{aligned} \tag{6}$$

We set the remainder as

$$R(r,m)=\frac{h^{2r+1}}{(2r+2)!}B_{2r+2}\left(\frac{1}{2}\right)\left(F^{(2r+1)}(b)-F^{(2r+1)}(a)\right)$$
$$-\frac{h^{2r+2}}{(2r+2)!}\int_0^1\hat{B}_{2r+2}\left(\frac{1}{2}-t\right)\left\{\sum_{k=0}^{m-1}F^{(2r+2)}(a+kh+th)\right\}dt\,. \tag{7}$$

When $t\in[0,1/2]$ then $1/2-t-\lfloor 1/2-t\rfloor=1/2-t-0=1/2-t\in[0,1/2]$. In a similar way, when $t\in\,]1/2,1]$, then $1/2-t-\lfloor 1/2-t\rfloor=1/2-t-(-1)=3/2-t\in\,[1/2,1[$. This is to say when $0\le t\le 1$ then $0\le x-\lfloor x\rfloor<1$. Because of

$$\frac{1}{h}\int_0^1\left\{h\sum_{k=0}^{m-1}F^{(2r+2)}(a+kh+th)\right\}dt=\frac{F^{(2r+1)}(b)-F^{(2r+1)}(a)}{h}$$

and (5) we get the limits for the integral (7) as

$$\left|\frac{h^{2r+2}}{(2r+2)!}\int_0^1\hat{B}_{2r+2}\left(\frac{1}{2}-t\right)\left\{\sum_{k=0}^{m-1}F^{(2r+2)}(a+kh+th)\right\}dt\right|\le\frac{(-1)^{r+1}B_{2r+2}\left(\frac{1}{2}\right)h^{2r+1}\left(F^{(2r+1)}(b)-F^{(2r+1)}(a)\right)}{(2r+2)!\left(1-2^{-(2r+1)}\right)}.$$

The remainder can be written now

$$|R(r,m)|\le\frac{(-1)^{r+1}B_{2r+2}\left(\frac{1}{2}\right)h^{2r+1}\left(F^{(2r+1)}(b)-F^{(2r+1)}(a)\right)}{(2r+2)!}\left|1\pm\left(\frac{2^{2r+1}}{2^{2r+1}-1}\right)\right|.$$

The maximum of the remainder will be obtained when

$$\left|1\pm\left(\frac{2^{2r+1}}{2^{2r+1}-1}\right)\right|\le 2+\frac{1}{2^{2r+1}-1}.$$

If now apply to the Harmonic Number, we have (note that $F^{(p)}(x)=(-1)^p p!x^{-(p+1)}$ and $a=1/2$, $b=1/2+n$, $h=1$)

$$\sum_{k=1}^{n}\frac{1}{k}=\sum_{k=0}^{n-1}\frac{1}{k+1}=\sum_{k=0}^{n-1}\frac{1}{\frac{1}{2}+k\cdot 1+\frac{1}{2}\cdot 1}$$
$$=\int_{\frac{1}{2}}^{n+\frac{1}{2}}\frac{1}{t}dt+\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k}\left\{-\left(n+\frac{1}{2}\right)^{-2k}+\left(\frac{1}{2}\right)^{-2k}\right\}+R(r,n) \tag{8}$$
$$=\log\left(n+\frac{1}{2}\right)+\log(2)-\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(n+\frac{1}{2}\right)^{2k}}+\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\frac{1}{2}\right)^{2k}}+R(r,n).$$

We can write now

$$\sum_{k=1}^{n}\frac{1}{k}-\log(n)=\log\left(\frac{n+\frac{1}{2}}{n}\right)+\log(2)-\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(n+\frac{1}{2}\right)^{2k}}+\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\frac{1}{2}\right)^{2k}}+R(r,n).$$

When $n\to\infty$ then the right side is an Euler-Mascheroni constant γ and therefore

$$\gamma=\log(2)+\frac{1}{2}\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{k\left(\frac{1}{2}\right)^{2k}}+R(r,\infty).$$

So, we can write (8) as

$$H_n=\gamma+\log\left(n+\frac{1}{2}\right)-\frac{1}{2}\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{k\left(n+\frac{1}{2}\right)^{2k}}+R(r,n)-R(r,\infty). \qquad (9)$$

Villarino (2008) showed that this is identical to

$$H_n=\frac{1}{2}\log(2m)+\gamma-\frac{1}{2}\sum_{k=1}^{r}\frac{\sum_{p=0}^{k}\binom{k}{p}\left(-\frac{1}{4}\right)^{k-p}B_{2p}\left(\frac{1}{2}\right)}{k\,(2m)^{k}}+\tilde{R}(r+1). \qquad (10)$$

where $m=n(n+1)/2$. This equation gives the factors of (3).

We do not give more details about this proof since it holds only for integer valued Harmonic numbers. Our goal is to demonstrate that Ramanujan's Expansion holds for real valued Harmonic numbers.

2. The Proof

Let us start with

$$H_\eta=\psi(\eta+1)+\gamma=\sum_{k=1}^{\infty}\frac{\eta}{k(k+\eta)}=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{1}{k}-\sum_{k=1}^{N}\frac{1}{k+\eta}\right\}. \qquad (11)$$

Because we will apply the Euler-Maclaurin formula to the last sum, we write

$$\sum_{k=1}^{N}\frac{1}{k+\eta}=\sum_{k=0}^{N-1}\frac{1}{\eta+k+1}=\sum_{k=0}^{N-1}\frac{1}{\eta+\frac{1}{2}+k\cdot 1+\frac{1}{2}\cdot 1}.$$

Setting $a=\eta+1/2$, $b=\eta+1/2+N$, $h=1$ and $F^{(p)}(x)=(-1)^p p!\,x^{-(p+1)}$ it gives

$$\sum_{k=1}^{N}\frac{1}{k+\eta}=\log\left(\eta+\frac{1}{2}+N\right)-\log\left(\eta+\frac{1}{2}\right)-\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\eta+\frac{1}{2}+N\right)^{2k}}+\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\eta+\frac{1}{2}\right)^{2k}}+R(r,N)$$

where the remainder has the upper bound

$$\left|R(r,N)\right|\le\frac{\left(2+\frac{1}{2^{2r+1}-1}\right)(-1)^{r+1}B_{2r+2}\left(\frac{1}{2}\right)}{(2r+2)\left(\eta+\frac{1}{2}\right)^{2r+2}}\left[1-\left(\frac{\eta+\frac{1}{2}}{\eta+\frac{1}{2}+N}\right)^{2r+2}\right]$$

Substituting these results back to the (11) we have

$$\begin{aligned}H_{\eta}&=\lim_{N\to\infty}\left\{\sum_{k=1}^{N}\frac{1}{k}-\log(N)+\log\left(\frac{N}{\eta+\frac{1}{2}+N}\right)+\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\eta+\frac{1}{2}+N\right)^{2k}}+R(r,N)\right\}+\log\left(\eta+\frac{1}{2}\right)-\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\eta+\frac{1}{2}\right)^{2k}}\\&=\gamma+\log\left(\eta+\frac{1}{2}\right)-\sum_{k=1}^{r}\frac{B_{2k}\left(\frac{1}{2}\right)}{2k\left(\eta+\frac{1}{2}\right)^{2k}}+R(r,\infty)\end{aligned}\qquad(12)$$

where

$$\left|R(r,\infty)\right|\le\frac{\left(2+\frac{1}{2^{2r+1}-1}\right)(-1)^{r+1}B_{2r+2}\left(\frac{1}{2}\right)}{(2r+2)\left(\eta+\frac{1}{2}\right)^{2r+2}}.\qquad(13)$$

We see that (12) is more general solution than (10) because it holds also for real values. Following the Villarino's proof (2008) it can be written as

$$\left(\eta+\frac{1}{2}\right)^{2}=\eta^{2}+\eta+\frac{1}{4}=2\frac{\eta(\eta+1)}{2}+\frac{1}{4}=2\mathfrak{m}\left(1+\frac{1}{4(2\mathfrak{m})}\right)$$

where $\mathfrak{M}=\eta(\eta+1)/2$. The logarithm gives now

$$\begin{aligned}\log\left(\eta+\frac{1}{2}\right)&=\frac{1}{2}\log(2\mathfrak{m})+\frac{1}{2}\log\left(1+\frac{1}{4(2\mathfrak{m})}\right)\\&=\frac{1}{2}\log(2\mathfrak{m})-\frac{1}{2}\sum_{k=1}^{r}\frac{\left(-\frac{1}{4}\right)^{k}}{k(2\mathfrak{m})^{k}}+\frac{1}{2}\sum_{k=r+1}^{\infty}\frac{(-1)^{k+1}\left(\frac{1}{4}\right)^{k}}{k(2\mathfrak{m})^{k}}.\end{aligned}\qquad(14)$$

This holds when $\eta^{2}+\eta-1/4>0$ or $\eta>(-1+\sqrt{2})/2\approx0.2071$. Similar way

$$\sum_{p=1}^{r}\frac{B_{2p}\left(\frac{1}{2}\right)}{2p\left(\eta+\frac{1}{2}\right)^{2p}}=\frac{1}{2}\sum_{p=1}^{r}\frac{B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{p}}\left(1+\frac{1}{4\left(2\mathfrak{m}\right)}\right)^{-p}.$$

Taking into account the Newton's Binomial series theorem (Abramowitz and Stegun, 1972)

$$\left(1+x\right)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^{n},\qquad \binom{\alpha}{n}=\frac{\alpha\left(\alpha-1\right)\left(\alpha-2\right)\cdots\left(\alpha-n+1\right)}{n!},$$

the last formula can be written as

$$\frac{1}{2}\sum_{p=1}^{r}\frac{B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{p}}\left(1+\frac{1}{4\left(2\mathfrak{m}\right)}\right)^{-p}=\frac{1}{2}\sum_{p=1}^{r}\frac{B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{p}}\sum_{k=0}^{\infty}\binom{-p}{k}\left(\frac{1}{4\left(2\mathfrak{m}\right)}\right)^{k}$$
$$=\frac{1}{2}\sum_{p=1}^{r}\sum_{k=0}^{\infty}\frac{\binom{k+p-1}{k}\left(-\frac{1}{4}\right)^{k}B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{p+k}}.$$

This holds when $\eta^{2}+\eta-1/4>0$ (the limit is the same with logarithm series (14). We take the first terms of the series and write them as a power of $q=k+p$

$$\frac{1}{2}\sum_{p=1}^{r}\sum_{k=0}^{\infty}\frac{\binom{k+p-1}{k}\left(-\frac{1}{4}\right)^{k}B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{p+k}}=\frac{1}{2}\sum_{q=1}^{r}\sum_{p=1}^{q}\frac{\binom{q-1}{q-p}\left(-\frac{1}{4}\right)^{q-p}B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{q}}$$
$$+\frac{1}{2}\sum_{p=1}^{r}\sum_{k=r+1-p}^{\infty}\frac{\left(-1\right)^{k}\binom{k+p-1}{k}\left(\frac{1}{4}\right)^{k}B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{p+k}}.\tag{15}$$

Adding the partial sums of r terms from (14) and (15) we get

$$\frac{1}{2}\sum_{k=1}^{r}\frac{\left(-\frac{1}{4}\right)^{k}}{k\left(2\mathfrak{m}\right)^{k}}+\frac{1}{2}\sum_{k=1}^{r}\sum_{p=1}^{k}\frac{\binom{k-1}{k-p}\left(-\frac{1}{4}\right)^{k-p}B_{2p}\left(\frac{1}{2}\right)}{p\left(2\mathfrak{m}\right)^{k}}=\frac{1}{2}\sum_{k=1}^{r}\sum_{p=0}^{k}\frac{\binom{k}{p}\left(-\frac{1}{4}\right)^{k-p}B_{2p}\left(\frac{1}{2}\right)}{k\left(2\mathfrak{m}\right)^{k}}$$

since

$$\binom{k-1}{k-p}=\binom{k-1}{p-1}=\frac{p}{k}\binom{k}{p}.$$

We have showed that

$$H_\eta = \frac{1}{2}\log(2\mathfrak{m}) + \gamma - \frac{1}{2}\sum_{k=1}^{r} \frac{(-1)^k \sum_{p=0}^{k} \binom{k}{p} \left(\frac{1}{4}\right)^{k-p} (-1)^p B_{2p}\left(\frac{1}{2}\right)}{k(2\mathfrak{m})^k} + \tilde{R}(r,\infty).$$

This is the same result than (10) but it holds more generally for all $\eta > (-1+\sqrt{2})/2,\ \eta \in \mathrm{R}_+$. Of course, this is asymptotic expansion for the Harmonic Number as $\eta \to \infty$, so it gives better estimations when η increase.

We must estimate the remainders yet. Since the series of the logarithm (14) is an alternating series, its remainder fulfils the condition

$$\left| \frac{1}{2}\sum_{k=r+1}^{\infty} \frac{(-1)^{k+1}\left(\frac{1}{4}\right)^k}{k(2\mathfrak{m})^k} \right| \le \frac{1}{2}\frac{\left(\frac{1}{4}\right)^{r+1}}{(r+1)(2\mathfrak{m})^{r+1}}.$$

In a similar way, the upper limit of the remainder of the series (15) is

$$\begin{aligned}
\left| \frac{1}{2}\sum_{p=1}^{r}\sum_{k=r+1-p}^{\infty} \frac{(-1)^k \binom{k+p-1}{k}\left(\frac{1}{4}\right)^k B_{2p}\left(\frac{1}{2}\right)}{p(2\mathfrak{m})^{p+k}} \right| &= \left| \frac{1}{2}\sum_{p=1}^{r} \frac{(-1)^p B_{2p}\left(\frac{1}{2}\right)}{p(2\mathfrak{m})^p} \sum_{k=r+1-p}^{\infty} (-1)^{k+p}\binom{k+p-1}{k}\left(\frac{1}{4(2\mathfrak{m})}\right)^k \right| \\
&\le \frac{1}{2}\frac{1}{(2\mathfrak{m})^{r+1}}\sum_{p=1}^{r}\frac{(-1)^p B_{2p}\left(\frac{1}{2}\right)}{p}\binom{r}{p-1}\left(\frac{1}{4}\right)^{r+1-p} \\
&= \frac{1}{2}\frac{\sum_{p=1}^{r}(-1)^p\binom{r+1}{p}\left(\frac{1}{4}\right)^{r+1-p} B_{2p}\left(\frac{1}{2}\right)}{(r+1)(2\mathfrak{m})^{r+1}}.
\end{aligned}$$

Thus, the total remainder is

$$\begin{aligned}
|\tilde{R}(r,\infty)| &\le \frac{\left(\frac{1}{4}\right)^{r+1}}{2(r+1)(2\mathfrak{M})^{r+1}} + \frac{\sum_{p=1}^{r}(-1)^p\binom{r+1}{p}\left(\frac{1}{4}\right)^{r+1-p}B_{2p}\left(\frac{1}{2}\right)}{2(r+1)(2\mathfrak{M})^{r+1}} + \frac{(-1)^{r+1}B_{2(r+1)}\left(\frac{1}{2}\right)}{2(r+1)(2\mathfrak{M})^{r+1}} + \frac{(-1)^{r+1}\left(1+\frac{1}{2^{2r+1}-1}\right)B_{2r+2}\left(\frac{1}{2}\right)}{2(r+1)(2\mathfrak{M})^{r+1}} \\
&= \frac{\sum_{p=0}^{r+1}(-1)^p\binom{r+1}{p}\left(\frac{1}{4}\right)^{r+1-p}B_{2p}\left(\frac{1}{2}\right)}{2(r+1)(2\mathfrak{M})^{r+1}} + \frac{(-1)^{r+1}B_{2r+2}\left(\frac{1}{2}\right)}{2(r+1)(2\mathfrak{M})^{r+1}}\left(1+\frac{1}{2^{2r+1}-1}\right).
\end{aligned}$$

If $\eta \ge 10$, when $2\mathfrak{M} = \eta(\eta+1) \ge 110$, and $r = 6$, then the error is less than 10^{-15}. In the case $\eta < 10$, we can find $k \in \mathrm{N}$ just that $\eta + k \in [10, 11[$. Using formula (2) we get

$$H_\eta = H_{\eta+k} - \sum_{j=1}^{k}\frac{1}{\eta+j}.$$

We showed empirically this expression in an earlier paper (Haarala and Orosco, 2016). With this algorithm, we can find any H_η, when $\eta \geq 0$. In fact, it is possible to find the values also in the cases $\eta < 0$, because (Abramowitz and Stegun, 1972)

$$\psi(\eta+1) = \psi(\eta) + \frac{1}{\eta},$$
$$\psi(-\eta+1) = \psi(\eta) + \pi\cot(\pi\eta).$$

gives the formula (without forgetting $H_\eta = \psi(\eta+1) + \gamma$)

$$H_{-\eta} = H_\eta - \frac{1}{\eta} + \pi\cot(\pi\eta).$$

Anyway, we need The Ramanujan's Expansion only in the case $\eta > 0$, when KS-3 is concerned.

3. Discussion

We showed the Ramanujan's Harmonic Number Expansion for the integer valued Harmonic numbers starting from the Euler-Maclaurin formula. This proof had seemed to be unknown after Ramanujan's death (1920) until now. Villarino gave the first proof of the Ramanujan's Expansion even though he did not found the connection to the Euler-Maclaurin formula and he also proved the result only for the integer valued Harmonic Numbers. Any way his work was an important key to find out our proof. We proved that the Ramanujan's Harmonic Number Expansion is also valid for more general Harmonic numbers . In the other words, the Ramanujan's Expansion is actually a Psi Function Expansion.

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