Cuadernos de Ingeniería. Nueva Serie.
Revista de la Facultad de Ingeniería de la
Universidad Católica de Salta, núm. 11, 2019
ISSN 2545-692X (papel) / 2542-7012 (on line): 55-66

Generalized proofs of the Kijko-Sellevoll functions

Pruebas generalizadas de las funciones Kijko-Sellevoll

Mika Haarala[1] y Lía Orosco[1,2]

Abstract

In this report, we prove the Kijko-Sellevoll formulae by mean of integration formulae. These proofs show that the Kijko-Sellevoll functions are also valid for real values. Besides, we generalized these integration formulae for the general moment, yielding a family of Kijko-Sellevoll functions related with the cumulative distribution function (CDF) of Gutenberg-Richter.

Keywords: Gutenberg-Richter distribution function – Kijko-Sellevoll function

Resumen

En este artículo probamos las fórmulas de Kijko-Sellevoll por medio de fórmulas integrales. Estas expresiones muestran que las funciones de Kijko-Sellevoll son válidas también para valores reales. Además, hemos encontrado una expresión para el momento, haciendo uso de estas fórmulas integrales. De esta manera hemos arribado a una familia de funciones de Kijko-Sellevoll relacionadas a la función de distribución acumulada (CSF) de Gutenberg-Richter.

Palabras clave: Función de distribución Gutenberg-Richter – función de Kijko-Sellevoll

1. Introduction

In earlier works (Haarala and Orosco, 2016a, 2016b) we were analyzing the double truncated Gutenberg-Richter distribution function

$$f(m)=\frac{\beta\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]},$$

which has cumulative distribution function (CDF)

Citar: Haarala, M.; Orosco, L. (2019). Generalized proofs of the Kijko-Sellevoll functions. *Cuadernos de Ingeniería. Nueva Serie*. [Salta - Argentina], núm. 11: 55-66.

[1] Instituto de Estudios Interdisciplinarios de Ingeniería (IESIING) – Facultad de Ingeniería – UCASAL
[2] Facultad de Ingeniería - U.N.Sa.

$$F_M\left(m \mid m_{\max}\right)=\begin{cases}0, & \text{for } m<m_{\min}, \\ \dfrac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]} & \text{for } m_{\min} \leq m<m_{\max}, \\ 1, & \text{for } m_{\max} \leq m.\end{cases} \tag{1}$$

The final CDF was defined as

$$F_{M_{(n)}}\left(m \mid m_{\max}\right)=\begin{cases}0, & \text{for } m<m_{\min}, \\ \left[F_M\left(m \mid m_{\max}\right)\right]^n & \text{for } m_{\min} \leq m<m_{\max}, \\ 1, & \text{for } m_{\max} \leq m.\end{cases}$$

Using the Kijko-Sellevoll function 1 (KS-1)

$$f_n^{KS-1}(x)=\sum_{k=1}^{\infty} \frac{\left(1-\exp[-x]\right)^k}{k+n} \tag{2}$$

or the Kijko-Sellevoll function 2 (KS-2)

$$f_n^{KS-2}(x)=\sum_{k=1}^{\infty} \frac{n}{k} \frac{\left(1-\exp[-x]\right)^k}{k+n}, \tag{3}$$

we wrote the expected value of the maximum $M_{(n)}$ of the artificial catalogue considered as

$$\begin{aligned} E\left(M_{(n)} \mid m_{\max}\right) &= m_{\max}-\frac{1}{\beta} f_n^{KS-1}\left(\beta\left(m_{\max}-m_{\min}\right)\right) \\ &= m_{\min}+\frac{1}{\beta} f_n^{KS-2}\left(\beta\left(m_{\max}-m_{\min}\right)\right). \end{aligned} \tag{4}$$

For the variance, we could define a Kijko-Sellevoll function 3 (KS-3) as

$$f_n^{KS-3}(x)=\sum_{k=2}^{\infty} \frac{2n}{2n+k}\left\{\sum_{j=1}^{k-1} \frac{1}{n+j}\right\} \frac{\left(1-\exp[-x]\right)^k}{n+k} \tag{5}$$

(this series could start also from k = 1 but in that case the first term is 0) so the variance can be written as

$$Var\left(M_{(n)} \mid m_{\max}\right)=\frac{1}{\beta^2} f_n^{KS-3}\left(\beta\left(m_{\max}-m_{\min}\right)\right). \tag{6}$$

Even though in previous reports $n \in \mathbb{N}$ was the number of events and we gave the proofs using integer value, the formulae (2)-(6) are also valid when $n = \eta \in \mathbb{R}_+$ as we mentioned.

2. Expected value

First, we see that

$$\begin{aligned}
&\frac{\partial}{\partial \mathfrak{m}}\left[\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{k+\eta}\right]\\
&=\frac{\partial}{\partial \mathfrak{m}}\left[\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k+\eta}}{k+\eta}\right]\\
&=\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{k=1}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k+\eta-1}\\
&=\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{k=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}\\
&=\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}.
\end{aligned} \tag{7}$$

The $\sum_{k=0}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}$ is a geometric series which gives $1/\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]$ when $-\log\left(2\right)\leq\beta\left(\mathfrak{m}-m_{\min}\right)<\infty$. The equation (7) gives an integration formula

$$\int\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}d\mathfrak{m}=\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{k+\eta}+C. \tag{8}$$

Let´s assume now that *M* is independently and identically distributed (iid) with

$$F_M\left(\mathfrak{m}\,|\,m_{\max}\right)=\begin{cases}0, & \text{for } \mathfrak{m}<m_{\min},\\ \left[\dfrac{1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^{\eta} & \text{for } m_{\min}\leq\mathfrak{m}<m_{\max},\\ 1, & \text{for } m_{\max}\leq\mathfrak{m},\end{cases} \tag{9}$$

for all $\eta \geq 0,\ \eta \in \mathrm{R}$. Then the expected value is

ISSN 2545-692X (papel) / 2542-7012 (on line): 55-66

$$
\begin{aligned}
E(M \mid m_{\max}) &= \int_{m_{\min}}^{m_{\max}} \mathfrak{m}\, dF_{M_{(n)}}(\mathfrak{m} \mid m_{\max}) \\
&= m_{\max} - \int_{m_{\min}}^{m_{\max}} F_{M_{(n)}}(\mathfrak{m} \mid m_{\max})\, d\mathfrak{m} \\
&= m_{\max} - \int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp[-\beta(\mathfrak{m}-m_{\min})]}{1-\exp[-\beta(m_{\max}-m_{\min})]} \right]^{\eta} d\mathfrak{m} \\
&= m_{\max} - \frac{\int_{m_{\min}}^{m_{\max}} [1-\exp[-\beta(\mathfrak{m}-m_{\min})]]^{\eta}\, d\mathfrak{m}}{[1-\exp[-\beta(m_{\max}-m_{\min})]]^{\eta}} \\
&= m_{\max} - \frac{1}{\beta}\sum_{k=1}^{\infty} \frac{(1-\exp[-\beta(m_{\max}-m_{\min})])^{k}}{k+\eta} \qquad \text{(KS-1)} \\
&= m_{\max} - \frac{1}{\beta}\sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{\eta}{k(k+\eta)} \right] (1-\exp[-\beta(m_{\max}-m_{\min})])^{k} \\
&= m_{\min} + \frac{1}{\beta}\sum_{k=1}^{\infty} \frac{\eta(1-\exp[-\beta(m_{\max}-m_{\min})])^{k}}{k(k+\eta)} \qquad \text{(KS-2)}
\end{aligned}
$$

as $\sum_{k=1}^{\infty} z^k / k = -\log(1-z),\ -1 \le z < 1,$ thus

$$
\beta^{-1}\sum_{k=1}^{\infty}(1-\exp[-\beta(m_{\max}-m_{\min})])^{k} / k = m_{\max}-m_{\min}
$$

when $-\log(2) \le \beta(m_{\max}-m_{\min}) < \infty$. These expected values can be also written as

$$
\begin{aligned}
E(\beta(m_{\max}-M) \mid m_{\max}) &= f_{\eta}^{KS-1}(\beta(m_{\max}-m_{\min})), \\
E(\beta(M-m_{\min}) \mid m_{\max}) &= f_{\eta}^{KS-2}(\beta(m_{\max}-m_{\min})),
\end{aligned}
$$

and they hold when $\eta \in \mathrm{R}_{+}$ and $-\log(2) \le \beta(m_{\max}-m_{\min}) < \infty$.

3. Variance

Before deriving the variance, we must find the integration formula as we have done above. We have

ISSN 2545-692X (papel) / 2542-7012 (on line): 55-66

$$\frac{\partial}{\partial \mathfrak{m}}\left[\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k+j}}{(\eta+k)(\eta+k+j)}\right]$$
$$=\frac{\partial}{\partial \mathfrak{m}}\left[\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta+k+j}}{(\eta+k)(\eta+k+j)}\right]$$
$$=\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta+k+j-1}}{(\eta+k)}$$
$$=\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta+k}}{\eta+k}\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\sum_{j=1}^{\infty}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{j-1}$$
$$=\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta+k}}{\eta+k}$$
$$=\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{\eta+k},$$

thus

$$\int\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{\eta+k}\,d\mathfrak{m}$$
$$=\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\frac{1}{\beta}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k+j}}{(\eta+k)(\eta+k+j)}+C.$$

when $\eta\in\mathrm{R}_{+}$ and $-\log(2)\leq\beta(m_{\max}-m_{\min})<\infty$. To get the variance we have

$$Var\left(M\mid m_{\max}\right)=E\left(M^{2}\mid m_{\max}\right)-\left[E\left(M\mid m_{\max}\right)\right]^{2}$$
$$=\frac{2}{\beta}\frac{\int_{m_{\min}}^{m_{\max}}\left[1-\exp\left[-\beta\left(m-m_{\min}\right)\right]\right]^{\eta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k}}{\eta+k}dm}{\left[1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right]^{\eta}}$$
$$-\left[\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{\eta+k}\right]^{2}$$
$$=\frac{2}{\beta^{2}}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k+j}}{(\eta+k)(\eta+k+j)}-\frac{1}{\beta^{2}}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k+j}}{(\eta+k)(\eta+j)}$$

$$
\begin{aligned}
&= \frac{1}{\beta^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left[\frac{2}{(\eta+k)(\eta+k+j)} - \frac{1}{(\eta+k)(\eta+j)}\right]\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k+j} \\
&= \frac{1}{\beta^2}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{\eta+j-k}{(\eta+k)(\eta+j)}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k+j}}{\eta+k+j} \\
&= \frac{1}{\beta^2}\sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{\eta+2j-k}{(\eta+k-j)(\eta+j)}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{\eta+k} \\
&= \frac{1}{\beta^2}\sum_{k=2}^{\infty}\frac{2\eta}{2\eta+k}\left[\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right]\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k}}{\eta+k},
\end{aligned}
$$

because

$$
\begin{aligned}
\sum_{j=1}^{k-1}\frac{\eta+2j-k}{(\eta+k-j)(\eta+j)} &= \sum_{j=1}^{k-1}\left[\frac{\eta}{(\eta+k-j)(\eta+j)} + \frac{2j-k}{(\eta+k-j)(\eta+j)}\right] \\
&= \sum_{j=1}^{k-1}\left[\frac{\eta}{2\eta+k}\left[\frac{1}{\eta+k-j}+\frac{1}{\eta+j}\right] + \frac{1}{\eta+k-j} - \frac{1}{\eta+j}\right] \\
&= \frac{\eta}{2\eta+k}\left[\sum_{j=1}^{k-1}\frac{1}{\eta+j}+\sum_{j=1}^{k-1}\frac{1}{\eta+j}\right] + \sum_{j=1}^{k-1}\frac{1}{\eta+j} - \sum_{j=1}^{k-1}\frac{1}{\eta+j} \\
&= \frac{2\eta}{2\eta+k}\sum_{j=1}^{k-1}\frac{1}{\eta+j}.
\end{aligned}
$$

Hence

$$
Var\left(\beta\left(m_{\max}-M\right)\mid m_{\max}\right) = Var\left(\beta\left(M-m_{\min}\right)\mid m_{\max}\right) = f_{\eta}^{KS-3}\left(\beta\left(m_{\max}-m_{\min}\right)\right).
$$

and they hold when $\eta \in \mathrm{R}_{+}$ and $-\log(2) \leq \beta(m_{\max} - m_{\min}) < \infty$.

4. General discussion about the KS functions

We can find the mathematical connection to the Incomplete Beta and Psi functions for the KS-1 and KS-2 functions. Let's start with

$$
E\left(\beta\left(m_{\max}-M\right)\mid m_{\max}\right) = \beta\int_{m_{\min}}^{m_{\max}}\left[\frac{1-\exp\left[-\beta\left(\mathfrak{M}-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\right]^{\eta} d\mathfrak{M}.
$$

If we change the variable defining $t = 1 - \exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]$, we obtain

$$\begin{aligned} E\left(\beta\left(m_{\max} - M\right) \mid m_{\max}\right) &= \frac{1}{t_{\max}^{\eta}} \int_0^{t_{\max}} \frac{t^{\eta}}{1-t} dt \\ &= \frac{\mathrm{B}_{t_{\max}}\left(\eta+1,0\right)}{t_{\max}^{\eta}} \\ &= \int_0^{t_{\max}} \left(\frac{t}{t_{\max}}\right)^{\eta} \frac{1}{1-t} dt \\ &= \mathfrak{B}_{t_{\max}}\left(\eta+1\right), \end{aligned}$$

where $t_{\max} = 1 - \exp[-\beta(m_{\max} - m_{\min})]$, and B is an Incomplete Beta function

$$\mathrm{B}_x\left(a,b\right) = \int_0^x \frac{t^{a-1}}{(1-t)^{b-1}} dt .$$

(Abramowitz and Stegun, 1972). There is no model for the function $\mathfrak{B}$, at least considering the literature we could access. Because of the difference between the functions B and $\mathfrak{B}$ is the nominator (while in the Incomplete Beta function B, the variable t in the nominator gets the values between 0 and $t_{\max}$, in the function $\mathfrak{B}$ it gets values between 0 and 1), we may call the function $\mathfrak{B}$ as an "Incomplete Beta function of Second Kind". Of course, when $m_{\max} \to \infty$ then $t_{\max} \to 1$ and

$$\mathfrak{B}_1\left(\eta+1\right) = \mathrm{B}_1\left(\eta+1,0\right) = \infty .$$

By the use of KS-2, we recall that

$$\begin{aligned} E\left(M \mid m_{\max}\right) &= m_{\max} - \int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right]^{\eta} d\mathfrak{M} \\ &= m_{\min} + \left(m_{\max} - m_{\min}\right) - \int_{m_{\min}}^{m_{\max}} \left[\frac{1-\exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right]^{\eta} d\mathfrak{M} \\ &= m_{\min} + \int_{m_{\min}}^{m_{\max}} 1 - \left[\frac{1-\exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right]^{\eta} d\mathfrak{M}, \end{aligned}$$

so we have

$$E\left(\beta\left(M - m_{\min}\right) \mid m_{\max}\right) = \beta \int_{m_{\min}}^{m_{\max}} 1 - \left[\frac{1-\exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}\right]^{\eta} d\mathfrak{M} .$$

As it was done above, we change the variable with $t = 1 - \exp\left[-\beta\left(\mathfrak{M} - m_{\min}\right)\right]$, thus

$$E\left(\beta\left(M - m_{\min}\right) \mid m_{\max}\right) = \int_0^{t_{\max}} \left(1 - \left(\frac{t}{t_{\max}}\right)^{\eta}\right) \frac{1}{1-t} dt .$$

Using the Incomplete Beta function like an example, we can define an Incomplete Psi function as

$$\psi_x(z)+\gamma=\int_0^x \frac{1-t^{z-1}}{1-t}dt$$

and the Incomplete Psi function of Second Kind as

$$\Psi_{t_{\max}}(\eta+1)+\gamma=\int_0^{t_{\max}}\left(1-\left(\frac{t}{t_{\max}}\right)^{\eta}\right)\frac{1}{1-t}dt\,.$$

where γ is an Euler-Mascheroni constant. In this case when $m_{\max}\rightarrow\infty$, which is to say $t_{\max}\rightarrow 1$, we get

$$\Psi_1(\eta+1)+\gamma=\psi_1(\eta+1)+\gamma=\psi(\eta+1)+\gamma=H_\eta\,, \qquad (10)$$

where H_η the is a generalized Harmonic Number of order 1, which can also be defined as

$$H_\eta=\begin{cases}\int_0^1 \frac{1-t^\eta}{1-t}dt, & \eta>0,\\ 0, & \eta=0.\end{cases}$$

The Psi function gives an interesting connection between the works of Aki (1965) and Utsu (1965). Utsu based his estimators on the Gamma function while Aki based on the Gutenberg-Richter distribution function with $m_{\max}=\infty$. We showed (Haarala and Orosco, 2016a, 2016b) that Aki's estimator is based on the Psi (Digamma) function (Abramowitz, Stegun, 1972)

$$\psi(z)=\frac{d}{dz}\log(\Gamma(z))$$

where we can observe the maximum likelihood method applied to the Gamma function. This is the connection between the Aki's and Utsu's estimators, showing their equality. Initially this estimator was found out by Utsu (1965) using Moment Estimator method and shortly after him, Aki (1965) found out the same estimator applying Maximum Likelihood method. Because of those estimators have no differences, we have named them in our works an Aki-Utsu estimator, being written as $\hat{b}_{AU}$ or $\hat{\beta}_{AU}=\hat{b}_{AU}\log(10)$.

The functions $\mathfrak{B}_{t_{\max}}(\eta+1)$ and $\Psi_{t_{\max}}(\eta+1)$ define the logarithm decomposition

$$\begin{aligned}\beta(m_{\max}-m_{\min})&=-\log[1-t_{\max}]\\ &=-\int_1^{1-t_{\max}}\frac{dt}{t}=\int_0^{t_{\max}}\frac{dt}{1-t}\\ &=\mathfrak{B}_{t_{\max}}(\eta+1)+\Psi_{t_{\max}}(\eta+1)+\gamma\end{aligned}$$

as

$$\frac{1}{1-t}=\left(\frac{t}{t_{\max}}\right)^{\eta}\frac{1}{1-t}+\left(1-\left(\frac{t}{t_{\max}}\right)^{\eta}\right)\frac{1}{1-t}$$

for all $\eta \geq 0$. In series, the last expression looks like

$$\frac{t_{\max}^{k}}{k}=\frac{t_{\max}^{k}}{k+\eta}+\frac{\eta\, t_{\max}^{k}}{k\left(k+\eta\right)}.$$

Because of those connections we proved to exist between the Psi and Beta functions and what we called Kijko-Sellevoll (KS) functions 1 and 2, we consider more appropriate to name them as KS functions. The reason is that we can see them like

$$f_{\eta}^{KS}\left(x\right)=\sum_{k=1}^{\infty}A_k\frac{\left(1-\exp[-x]\right)^{k}}{\eta+k}, \tag{11}$$

where

$$A_k=1, \qquad \text{for KS-1 } (2),$$

$$A_k=\frac{\eta}{k}, \qquad \text{for KS-2 } (3),$$

$$A_k=\begin{cases}0, & k=1,\\ \dfrac{2\eta}{2\eta+k}\displaystyle\sum_{j=1}^{k-1}\frac{1}{\eta+j}, & k\geq 2,\end{cases} \qquad \text{for KS-3 } (5).$$

At it happens with (7) and (8) it is possible to find a general integration formula

$$\int\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_{n-1}=1}^{\infty}\left\{\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k_1+k_2+\cdots+k_{n-1}}}{\left(\eta+k_1\right)\left(\eta+k_1+k_2\right)\cdots\left(\eta+k_1+k_2+\cdots+k_{n-1}\right)}\right\}d\mathfrak{m}$$

$$=\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\frac{1}{\beta}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k_1+k_2+\cdots+k_n}}{\left(\eta+k_1\right)\left(\eta+k_1+k_2\right)\cdots\left(\eta+k_1+k_2+\cdots+k_n\right)}+C.$$

The moments of the Gutenberg – Richter distribution function are written as:

$$
\begin{aligned}
E\left(M^{n} \mid m_{\max}\right) &= \int_{m_{\min}}^{m_{\max}} \mathfrak{m}^{n}\, dF_{M}\left(\mathfrak{m} \mid m_{\max}\right) = m_{\max}^{n} - n\int_{m_{\min}}^{m_{\max}} \mathfrak{m}^{n-1} F_{M}\left(\mathfrak{m} \mid m_{\max}\right) d\mathfrak{m} \\
&= m_{\max}^{n} - \frac{n\int_{m_{\min}}^{m_{\max}} \mathfrak{m}^{n-1}\left[1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right]^{\eta} d\mathfrak{m}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{\eta}} \\
&= m_{\max}^{n} - \frac{n\, m_{\max}^{\;n-1}}{\beta}\sum_{k_1=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k_1}}{\eta+k_1} \\
&\quad + \frac{\frac{n(n-1)}{\beta}\int_{m_{\min}}^{m_{\max}} \mathfrak{m}^{n-2}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\sum_{k_1=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k_1}}{\eta+k_1} d\mathfrak{m}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{\eta}} \\
&= m_{\max}^{n} - \frac{n\, m_{\max}^{\;n-1}}{\beta}\sum_{k_1=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k_1}}{\eta+k_1} \\
&\quad + \frac{n(n-1)m_{\max}^{n-2}}{\beta^{2}}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k_1+k_2}}{\left(\eta+k_1\right)\left(\eta+k_1+k_2\right)} \\
&\quad - \frac{\frac{n(n-1)(n-2)}{\beta^{2}}\int_{m_{\min}}^{m_{\max}} \mathfrak{m}^{n-3}\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{\eta}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(\mathfrak{m}-m_{\min}\right)\right]\right)^{k_1+k_2}}{\left(\eta+k_1\right)\left(\eta+k_1+k_2\right)} d\mathfrak{m}}{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{\eta}} \\
&= \cdots .
\end{aligned}
$$

Hence

$$
\begin{aligned}
E\left(M^{n} \mid m_{\max}\right) &= m_{\max}^{n} + \sum_{p=1}^{n}\left\{(-1)^{p}\frac{n(n-1)\cdots(n-p+1)m_{\max}^{n-p}}{\beta^{p}}\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_p=1}^{\infty}\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{k_1+k_2+\cdots+k_p}}{\left(\eta+k_1\right)\left(\eta+k_1+k_2\right)\cdots\left(\eta+k_1+k_2+\cdots+k_p\right)}\right\} \\
&= m_{\max}^{n} + \sum_{p=1}^{n}\left\{(-1)^{p}\frac{n(n-1)\cdots(n-p+1)m_{\max}^{n-p}}{\beta^{p}}\sum_{j=p}^{\infty}\left[\sum_{k_1+\cdots+k_p=j}\frac{1}{\left(\eta+k_1\right)\cdots\left(\eta+k_1+\cdots+k_{p-1}\right)}\right]\frac{\left(1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]\right)^{j}}{\eta+j}\right\},
\end{aligned}
$$

where the sum $\sum_{k1+\cdots+kp=j}$ pass all the possible combinations of integer $k_r > 0$, such that $r=1,2,\ldots,p$, their sum is $k_1+\cdots+k_p=j$.

If we define as

$$
A_j = \begin{cases} 0, & j < p, \\ \displaystyle\sum_{k_1+\cdots+k_p=j}\frac{1}{\left(\eta+k_1\right)\cdots\left(\eta+k_1+\cdots+k_{p-1}\right)}, & j \ge p, \end{cases}
$$

then we get a family of KS functions, which are related with the CDF of Gutenberg-Richter .

5. Conclusion

We proved that the Kijko-Sellevoll functions are valid for non-negative real values. We showed how the KS-1 and KS-2 are related to the Incomplete Beta function of Second Kind and Incomplete Psi function of Second Kind function, respectively. Applying these results, we get a family of Kijko-Sellevoll functions related with the CDF of Gutenberg-Richter.

References

Abramowitz, M., and Stegun, I.A. (1972), *Handbook of mathematical functions*, 10th ed., Dover Publ., New York.

Aki, K. (1965). Maximum likelihood estimate of *b* in the formula $\log N = a - bM$ and its confidence limits, *Bull. Earthquake Res. Inst., Tokyo Univ. 43*, 237-239.

Haarala, M. and Orosco, L. (2016a). Analysis of Gutenberg-Richter *b*-value and m_{max}. Part I: Exact solution of Kijko-Sellevoll estimator m_{max}, *Cuadernos de Ingeniería*. *Nueva Serie.* Publicaciones Académicas Fac. Ingeniería, Universidad Católica de Salta, vol. 9, 2016, p 51-78. http://www.ucasal.edu.ar/eucasa/documentos/174-cuaderno-ingenieria-9.pdf (Last access: 12.04.2017).

Haarala, M. and Orosco, L. (2016b). Analysis of Gutenberg-Richter *b*-value and m_{max}. Part II: Estimators for *b*-value and exact variance, *Cuadernos de Ingeniería*. *Nueva Serie.* Publicaciones Académicas Fac. Ingeniería, Universidad Católica de Salta, vol. 9, 2016, p 79-106. http://www.ucasal.edu.ar/eucasa/documentos/174-cuaderno-ingenieria-9.pdf (Last access: 12.04.2017).

Utsu, T. (1965). A method for determining the value of *b* in a formula $\log n = a - bM$ showing the magnitude-frequency relation for earthquakes, *Geophys. Bull. Hokkaido Univ. 13*, 99-113.

Recibido: octubre de 2018
Aceptado: julio de 2019