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The quartile functions for the Generalized Gutenberg-Richter distribution

Las funciones del cuartil para la Distribución de la Función de Gutenberg-Richter generalizada

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Abstract
This study provides quartile and quartile density functions for the Generalized Gutenberg-Richter function. The quartile function provides a way to write a random number generator for the Gutenberg-Richter distributed data. Moreover, we also show the general limitations of the Gutenberg-Richter parameters β, $m_{\max}$ and $m_{\min}$ in this contribution.

Keywords: Gutenberg-Richter distribution, quartile function, quartile density function.

Resumen
En este trabajo se proponen funciones de cuartiles y funciones de densidad de cuartiles para la función de Gutenberg-Richter generalizada. La función cuartil permite escribir un generador de números aleatorios para los datos distribuidos de Gutenberg-Richter. Además, en esta contribución, demostramos las limitaciones generales de los parámetros β, $m_{\max}$ y $m_{\min}$ en este trabajo.

Palabras clave: Distribución de Gutenberg-Richter, función cuartil, función de densidad de cuartil

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1. Introduction

The probability density function (PDF) of the General Gutenberg-Richter (GGR) distribution function, also known as a double truncated exponential distribution, is given by

$$f(m)=\begin{cases}\dfrac{\beta\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}, & \text{for } m_{\min}\leq m\leq m_{\max} \text{ and } \beta\neq 0,\\ \dfrac{1}{m_{\max}-m_{\min}}, & \text{for } m_{\min}\leq m\leq m_{\max} \text{ and } \beta=0,\\ 0, & \text{for } m\notin\left[m_{\min},m_{\max}\right]\end{cases} \quad (1)$$

with cumulative distribution function (CDF)

$$\begin{cases}0, & \text{for } m<m_{\min},\\ \dfrac{1-\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}, & \text{for } m_{\min}\leq m<m_{\max}\ \wedge\ \beta\neq 0,\\ \dfrac{m-m_{\min}}{m_{\max}-m_{\min}} & \text{for } m_{\min}\leq m<m_{\max}\ \wedge\ \beta=0,\\ 1, & \text{for } m\geq m_{\max},\end{cases} \quad (2)$$

where $-\infty\leq\beta\leq\infty$ and $-\infty\leq m_{\min}\leq m_{\max}\leq\infty$ (Haarala, 2021). The distribution of the maximum of the GGR is

$$F_{M_\eta}(m)=\begin{cases}0, & \text{for } m<m_{\min},\\ \left[F_M(m)\right]^{\eta} & \text{for } m_{\min}\leq m<m_{\max},\\ 1, & \text{for } m_{\max}\leq m,\end{cases} \quad (3)$$

where $\eta\in\mathbb{R}_+$.

An important difference between the classical definition and the GGR is for the particular case $\beta=0$. The classical definition assumes that $\beta>0$, but because the estimators can be calculated at any $\eta\in\mathbb{R}_+$ it is necessarily to consider the distribution at the case $\beta\leq 0$ (Haarala, 2021). It should be noted that the classical form of the distribution (1) holds also for $\beta<0$, but it has a discontinuity at $\beta=0$. That is why the classical definition needs the Uniform distribution at $\beta=0$.

2. Quartile and quartile density functions

The case when β is non-zero

Because the CDF shown in equation (2) is invertible, the quartile function can be obtained for the case in which $\beta \neq 0$ as shown in the following expression

$$Q(p) = m_{\min} - \frac{1}{\beta}\log\left(1 - \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)p\right),$$

where $p \in [0,1]$. A quartile density function ($q = Q'$) can yield the following expression

$$q(p) = \frac{1}{\beta}\frac{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}{1 - \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)p}.$$

Earlier work (Haarala, 2021) has shown that if $\beta\left(m_{\max} - m_{\min}\right) \to \infty$, it is possible to find the following equation

$$\eta\sum_{k=1}^{\infty}\frac{\left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right)^k}{k(k+\eta)} = \sum_{k=1}^{\infty}\frac{\eta}{k(k+\eta)} = H_\eta. \tag{4}$$

where H_η is a General Harmonic number (Abramowitz and Stegun, 1972; Haarala and Orosco, 2016). If $0 < \beta < \infty$ and $m_{\max} - m_{\min} \to \infty$, then

$$E\left(M_{(\eta)}\right) = m_{\min} + \frac{H_\eta}{\beta} \quad \Leftrightarrow \quad m_{\min} = E\left(M_{(\eta)}\right) - \frac{H_\eta}{\beta}.$$

If the expected value is bounded, as they are in practical cases, the minimum $m_{\min}$ is bounded as well.

If the β is unbounded, the limit of series (4) is still H_η when $\beta\left(m_{\max} - m_{\min}\right) \to \infty$. For this case, the difference $m_{\max} - m_{\min}$ can be bounded or unbounded. Because the General Harmonic number is bounded $0 < H_\eta < \infty$ when $0 < \eta < \infty$ (of course, $H_\eta \to \infty$ as $\eta \to \infty$), the expected value is then equal to

$$E\left(M_\eta\right) = m_{\min}$$

for all $0 < \eta < \infty$. Again, because the expected value is assumed to be bounded, the minimum must be bounded. Thus, the limits can be written as $-\infty < m_{\min} \leq m_{\max} \leq \infty$, when $\beta > 0$. Moreover, if $E\left(M_{(\eta_1)}\right) \neq E\left(M_{(\eta_2)}\right)$, then $0 < \beta < \infty$.

It can be shown that for the negative values of β, that

$$\frac{\sum_{k=1}^{\infty}\frac{(-1)^k}{k+\eta}+\sum_{k=0}^{\infty}\frac{(-1)^k\left[\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{-k+\eta}-1\right]}{k-\eta}}{\left(\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]-1\right)^{\eta}}=-\frac{1}{\eta}, \tag{5}$$

when $\beta\left(m_{\max}-m_{\min}\right)\to-\infty$ (Haarala, 2021). Now, if $-\infty<\beta<0$ and $m_{\max}-m_{\min}\to\infty$, then

$$E\left(M_{(\eta)}\right)=m_{\max}+\frac{1}{\beta\eta} \quad\Leftrightarrow\quad m_{\max}=E\left(M_{(\eta)}\right)-\frac{1}{\beta\eta}. \tag{6}$$

It follows that the expected value $E\left(M_{(\eta)}\right)$ is bounded if and only if $m_{\max}$ is also bounded. Moreover, because η^{-1} is bounded for all $\eta>0$, then

$$E\left(M_{(\eta)}\right)=m_{\max}$$

when $\beta\to-\infty$. In this case, the maximum $m_{\max}$ is bounded because the estimators of expected values are assumed to be bounded. Thus, the limits are $-\infty\le m_{\min}\le m_{\max}<\infty$, when $\beta<0$. And if $E\left(M_{(\eta_1)}\right)\neq E\left(M_{(\eta_2)}\right)$, then $-\infty<\beta<0$.

The case where β is zero

When $\beta=0$, then the quartile function of equation (2) is

$$Q(p)=m_{\min}+\left(m_{\max}-m_{\min}\right)p,$$

where $p\in[0,1]$. The quartile density function yields the following expression

$$q(p)=m_{\max}-m_{\min}.$$

Previous results have shown that for the case of a Uniform distribution (Haarala, 2021), the bounds are

$$\begin{aligned} m_{\max}&=\frac{\eta_1+1}{\eta_1-\eta_2}E\left(M_{(\eta_1)}\right)-\frac{\eta_2+1}{\eta_1-\eta_2}E\left(M_{(\eta_2)}\right),\\ m_{\min}&=-\eta_2\frac{\eta_1+1}{\eta_1-\eta_2}E\left(M_{(\eta_1)}\right)+\eta_1\frac{\eta_2+1}{\eta_1-\eta_2}E\left(M_{(\eta_2)}\right).\end{aligned} \tag{7}$$

According to relation (6), if the estimators for the expected values are bounded, then $E\left(M_{(\eta_1)}\right)$ and $E\left(M_{(\eta_2)}\right)$ are also bounded. This indicates that the maximum $m_{\max}$ and minimum $m_{\min}$ are bounded. Anyway, if the maximum or the minimum is unbounded, also one of the expected values and its estimator must be unbounded. In the case of $\beta = 0$, the limits can be set as $-\infty < m_{\min} \leq m_{\max} < \infty$. Moreover, if $E\left(M_{(\eta_1)}\right) = E\left(M_{(\eta_2)}\right)$, then

$$m_{\min} = m_{\max} = E\left(M_{(\eta_1)}\right).$$

Let's assume that all generalized estimators are bounded for $n \in \{1, 2, \ldots, N\}$, where N is a size of catalog. Let's suppose that using these estimators there are only positive estimates for the β_n. We cannot draw any conclusion about the boundedness (or unboundedness) of the upper limit $m_{\max}$ from the fact that β_n is strictly positive for all n. However, it does indicate that the lower limit $m_{\min}$ is bounded.

In a similar way a strictly negative β_n does not necessarily indicate the boundedness of the lower limit $m_{\min}$, but is does indicate a bounded upper limit $m_{\max}$.

The quartile functions

Considering the results in the previous sections, the quartile function can be defined as

$$Q(p) = \begin{cases} m_{\min} - \dfrac{1}{\beta} \log\left(1 - \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right) p\right), & \begin{cases} \text{for } \beta > 0, & -\infty < m_{\min} \leq m_{\max} \leq \infty, \\ \text{for } \beta < 0, & -\infty \leq m_{\min} \leq m_{\max} < \infty, \end{cases} \\ m_{\min} + \left(m_{\max} - m_{\min}\right) p, & \quad \text{for } \beta = 0, \quad -\infty < m_{\min} \leq m_{\max} < \infty, \end{cases} \tag{8}$$

which has a quartile density function

$$q(p) = \begin{cases} \dfrac{1}{\beta} \dfrac{1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]}{1 - \left(1 - \exp\left[-\beta\left(m_{\max} - m_{\min}\right)\right]\right) p}, & \begin{cases} \text{for } \beta > 0, & -\infty < m_{\min} \leq m_{\max} \leq \infty, \\ \text{for } \beta < 0, & -\infty \leq m_{\min} \leq m_{\max} < \infty, \end{cases} \\ m_{\max} - m_{\min}, & \quad \text{for } \beta = 0, \quad -\infty < m_{\min} \leq m_{\max} < \infty. \end{cases}$$

It is not difficult to find the quartile function also for the maximum distribution shown in equation (4). One can replace p with $p^{1/\eta}$ in the quartile function. For the quartile density function this means that $q(p)$ is replaced by $q(p)/\eta$ and $q(p)\,p^{1/\eta - 1}/\eta$ for cases when $\beta \neq 0$ and $\beta = 0$, respectively.

In the in the appendix is given an example code to obtain GGR distributed random numbers.

3. Delta distribution

When $m_{\max} - m_{\min} \to 0$ the distribution tends to a Dirac Delta distribution. Taking as definition of the Delta distribution the following two equalities:

$$\delta(x) = \begin{cases} +\infty, & \text{if } x = 0, \\ 0, & \text{if } x \in \mathbb{R} \setminus \{0\}, \end{cases}$$

and

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

The cumulative distribution of the Dirac Delta distribution is the Heaviside step function (or unit step function), which is defined as

$$H(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

For our purposes, we need to shift the Delta distribution to the value $c = m_{\min} = m_{\max}$, which is done as $\delta(x - c)$, and the corresponding Heaviside step function is $H(x - c)$.

The interpretation of having a Delta distribution is that "random" samples taken from the distribution have their outcomes totally certain - that is, they are deterministic and not random at all.

For the case of the CDF of the GGR, the following inequalities hold

$$0 \leq \frac{1 - \exp\left[-\beta(m - m_{\min})\right]}{1 - \exp\left[-\beta(m_{\max} - m_{\min})\right]} < 1, \qquad \text{for all } m \in [m_{\min}, m_{\max}[\wedge \beta \neq 0,$$

$$0 \leq \frac{m - m_{\min}}{m_{\max} - m_{\min}} < 1, \qquad \text{for all } m \in [m_{\min}, m_{\max}[\wedge \beta = 0.$$

When $m_{\max} - m_{\min} \to 0$, the limit of the CDF (2) yields

$$F_M(m) = H(x - c).$$

Because,

$$\int_{-\infty}^{\infty} f(m)\, dm = \int_{m_{\min}}^{m_{\max}} f(m)\, dm = 1$$

for each $m_{\min} < m_{\max}$, so

$$\lim_{m_{\min} \to m_{\max}} \int_{m_{\min}}^{m_{\max}} f(m)\, dm \to 1 \quad \left[= \int_{c}^{c} f(m)\, dm \right]$$

At the same time, it holds for the PDF (1)

$$f(m)=\begin{cases}\dfrac{\beta\exp\left[-\beta\left(m-m_{\min}\right)\right]}{1-\exp\left[-\beta\left(m_{\max}-m_{\min}\right)\right]}\to\infty, & \text{for } m_{\min}\le m\le m_{\max} \text{ and } \beta\neq 0,\\ \dfrac{1}{m_{\max}-m_{\min}}\to\infty, & \text{for } m_{\min}\le m\le m_{\max} \text{ and } \beta=0,\\ 0, & \text{for } m\notin\left[m_{\min},m_{\max}\right]\end{cases}$$

as $m_{\max}-m_{\min}\to 0+$. Hence,

$$f(m)=\delta(x-c).$$

The expected value for the maximum for the case is

$$E\left(M_\eta\right)=\int_{m_{\min}}^{m_{\max}} m\, dF_{M_\eta}(m)=m_{\max}-\int_{m_{\min}}^{m_{\max}} F_{M_\eta}(m)\,dm\to m_{\max}\quad\left[=m_{\min}\right]$$

for all $\eta>0$. If $E\left(M_{(\eta_1)}\right)\neq E\left(M_{(\eta_2)}\right)$, then it must be $m_{\min}<m_{\max}$.

4. Examples

Let $m_1,m_2,\ldots,m_n$ be magnitudes in random order. Then the ordered sequence is $m_{(1)}\le m_{(2)}\le\ldots\le m_{(n)}$. The index in parentheses indicates that the events are ordered. For example, the minimum value is $m_{(1)}=\min\left(m_1,m_2,\ldots,m_n\right)$ and the maximum value from all events is $m_{(n)}=\max\left(m_1,m_2,\ldots,m_n\right)$.

Figure 1 presents the ordered sets of randomly generated magnitudes with different b-values (note: $\beta=b\log(10)$). We can see how the curves of positive b values are similar to the negative ones if they are rotated 180 degrees around the center point.

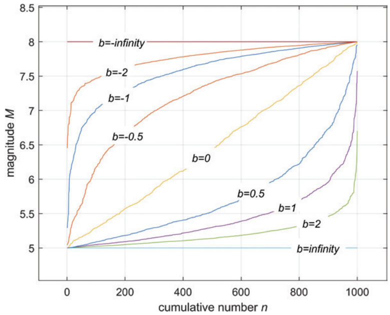

Figura 1. Cumulative plot of random samples of 1000 events from the GGR for different b-values

Figure 2 plots out the idea, how the ordered random events behave when they have same positive b values with a fixed minimum m_{min}, but the maximum m_{max} varies. Figure 3 shows the same idea, but for the fixed negative b, for fixed maximum m_{max} and using different minimum m_{min} values.

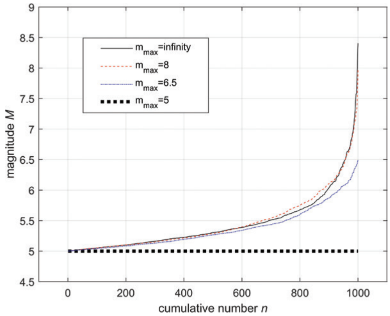

Figura 2. Cumulative plot with varying values of m_{max} and b-value fixed at b=1, and m_{min}= 5.

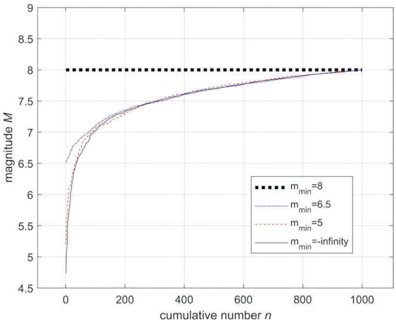

Figura 3. Cumulative plot with varying values of m_{min} and b-value fixed at b=-1, and m_{max}= 8.

5. Conclusions

This contribution gives the solution for the quartile functions and generate the Gutenberg-Richter distributed data. It shows also, as the estimators of real earthquake data gives at least two unequal estimates and the data is related with positive β, it holds $0 < \beta < \infty$ and $-\infty < m_{\min} < m_{\max} \leq \infty$. A corner magnitude m_c is not necessarily a minimum. We can say only $-\infty < m_{\min} \leq m_c < m_{\max} \leq \infty$. Moreover, it was shown that the general estimator for the β can change the sign only if the upper and lower limits $m_{\max}$ and $m_{\min}$ are bounded.

References

Abramowitz, M., and Stegun, I. A. (1972). *Handbook of mathematical functions*, Dover Publ.

Haarala, M., and Orosco, L. (2016). Analysis of Gutenberg-Richter *b*-value and m_{max}. Part I: Exact solution of Kijko-Sellevoll estimator m_{max}, *Cuadernos de Ingeniería. Nueva Serie*, (9), 51-78. http://revistas.ucasal.edu.ar/index.php/CI/article/view/145

Haarala, M. (2021). Analysis of Gutenberg-Richter b-value and m_{max}. Part III: Non-positive Gutenberg-Richter b-value, *Cuadernos de Ingeniería. Nueva Serie* (13), 45-84. https://doi.org/10.53794/ci.v13iXIII.348

Appendix

We give an example code for MATLAB to generate the GGR distributed random numbers using the quartile function (8). Because of numerical stability, the uniform distribution is used when $|\beta| < 10^{-8}$ instead of $\beta = 0$. This limit is given empirically.

```
function C = GRdistribution(beta,mmax,mmin,m,n)
%
% Input:
%   beta  = beta-value, scalar.
%   mmax  = maximum value, scalar.
%   mmin  = minimum value, scalar.
%   m,n   = size of output.
%

%  Authors: Mika Haarala Orosco, Petrus Vermeulen


% Check the arguments
if ~isscalar(beta) || ~isscalar(mmax) || ~isscalar(mmin) ||...
  ~isscalar(m)    || ~isscalar(n)
   error('All input arguments must be scalars.')
end


% Generate the random data.
p = rand(m,n);
if abs(beta) < 10^-8
   if isinf(mmax) || isinf(mmin)
      error('Limits mmax and mmin must be bounded for b=0.')
   end
   C = mmin + (mmax-mmin)*p;
else
   if beta > 0 && isinf(mmin)
      error('Lower limit mmin must be bounded for b>0.')
   elseif beta < 0 && isinf(mmax)
      error('Upper limit mmax must be bounded for b<0.')
   end
   C = mmin - log( 1 - (1-exp(-beta*(mmax-mmin)))*p ) / beta;
end
```

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